The First Order

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The questions depend on the **context**.

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Not all questions make sense everywhere.

Models: The context of first-order logic

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Meaning

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- **Functions:** Takes some elements from our universe and spits out another.
- **Relations:** Tell us if some elements from our universe are linked in some way.

$\text{Th}(\mathbb{R}, 0, 1, +, \times, <)$

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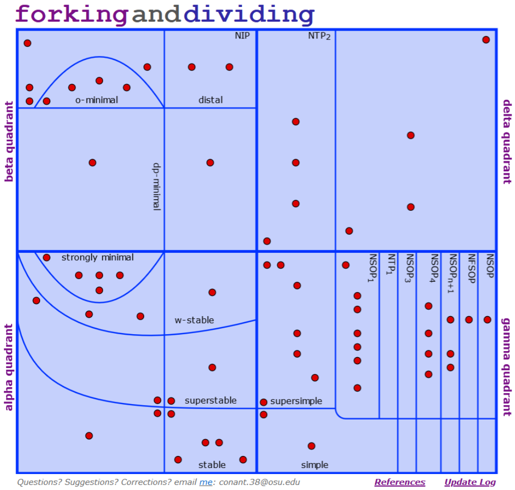
The *theory of* $\mathbb{R}$, $\mathrm{Th}(\mathbb{R}) := \{\varphi \mid \mathbb{R} \models \varphi\}$

Dividing lines



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A model $\mathcal{M}$ is *unstable* if we can find a formula $\varphi(x, y)$ such that $\mathcal{M} \vDash$"φ is a linear order". It is *stable* if we can't.

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$\varphi(x, y)$:	"$(\exists z)(z \neq 0 \wedge x + (z \times z) = y)$"

Then if $a, b \in \mathbb{R}$, $\mathbb{R} \vDash \varphi(a, b)$ if and only if $a < b$. Therefore $\mathrm{Th}(\mathbb{R}, +, \times, 0)$ is *unstable*.

Picking out models

Definition

Let $\mathbb{R}_{\text{alg}}$ be the set of *real algebraic numbers*: The (real!) solutions to polynomials $p_n X^n + p_{n-1} X^{n-1} + \cdots + p_1 X + p_0 = 0$, where $p_0, p_1, \ldots, p_n \in \mathbb{Q}$.

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Then $(\mathbb{R}, +, \times, 0, 1)$ and $(\mathbb{R}_{\text{alg}}, +, \times, 0, 1)$ are *elementarily equivalent*. $\text{Th}(\mathbb{R}) = \text{Th}(\mathbb{R}_{\text{alg}})$

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(finally)

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What *I* do

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- Take a familiar structure
- Create a model with a **language**
- Discern what the model believes
- Classify the theory
- Get information about similar structures

- The real numbers $\mathbb{R}$
- $(\mathbb{R}; +, \times, 0, 1, <)$
- $\mathrm{Th}(\mathbb{R})$, the **theory**
- $\mathrm{Th}(\mathbb{R})$ is *o*-minimal
- Definable sets in any "real-closed field" are made of "cells".