The Axiom of Extendable Choice

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The Partition Principle

Notation (Injections and surjections)

Given sets X and Y, write: $|X| \leqslant |Y|$ if there is an injection $X \to Y$; $|X| \leqslant^* |Y|$ if there is a surjection $Y \to X$, or if $X = \emptyset$; and $|X| = |Y|$ if there is a bijection $X \to Y$.

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Suppose that $f\colon Y \to X$ is a surjection. Let $g\colon X \to Y$ be such that $g(x) \in f^{-1}(x)$ for all $x \in X$. Then g is an injection. $\square$

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Question: Partition Problem

Does PP imply AC?

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> *Only later, when we better understand transfinite cardinal numbers, will the truth become clear: for all cardinal numbers* $\mathfrak{a}$ *and* $\mathfrak{b}$*, either* $\mathfrak{a} = \mathfrak{b}$*,* $\mathfrak{a} < \mathfrak{b}$*, or* $\mathfrak{a} > \mathfrak{b}$*.* [Can32, p. 285]

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> *Only later, when we better understand transfinite cardinal numbers, will the truth become clear: for all sets* X *and* Y*, either* $|X| \leqslant |Y|$ *or* $|X| \geqslant |Y|$. [Can32, p. 285]

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For all X there is an ordinal α such that $|\alpha| \nleq |X|$.

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AC *is equivalent to the statement that, for all* X *and* Y*,* $|X| \leqslant |Y|$ *or* $|X| \geqslant |Y|$*.*

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Proof.

Let X be a set, and α an ordinal such that $|\alpha| \not\leqslant |X|$. Then $|X| \leqslant |\alpha|$. □

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Note that $\aleph^*(X)$ exists, is $\aleph_\alpha$ for some α, and $\aleph(X) \leqslant \aleph^*(X)$.

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Theorem (Pincus, [Pel78; Pin69] resp.)

1. $\aleph = \aleph^*$ *is equivalent to* $\mathrm{AC}_{\mathrm{WO}}$*.*
2. *If* ZF *is consistent, then so is* $\mathrm{ZF} + \mathrm{AC}_{\mathrm{WO}} + \neg\mathrm{PP}$ *(hence* $\neg\mathrm{AC}$*).*

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($\Longleftarrow$). Let $\alpha \in \mathrm{Ord}$ be such that $|\alpha| \leqslant^* |X|$, witnessed by $f : X \to \alpha$.

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Proof.

$(\Longleftarrow)$. Let $\alpha \in \mathrm{Ord}$ be such that $|\alpha| \leqslant^* |X|$, witnessed by $f\colon X \to \alpha$. By $\mathrm{AC}_{\mathrm{WO}}$, $\{f^{-1}(\{\beta\}) \mid \beta < \alpha\}$ has a choice function $g\colon \alpha \to X$, which is an injection. Hence, $\alpha < \aleph^*(X) \implies \alpha < \aleph(X)$. □

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If this is well-defined then we're done. Otherwise, $f``\lambda \subseteq \bigcup_{\beta<\alpha} D_\beta$ for some β. □

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The ordinary partition problem

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Does

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Theorem (Pincus)

It is the case that

"For all $\underline{X \in \mathrm{WO}}$ and Y, $|X| \leqslant^* |Y|$, then $|X| \leqslant |Y|$"

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A Pincus-style characterisation

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Given a model $M \models \mathsf{ZF}$, the *Hartogs–Lindenbaum spectrum* of M is the class

$$\mathrm{Spec}_{\aleph}(M) := \{\langle \aleph(X), \aleph^*(X)\rangle \mid X \in M\}.$$

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Theorem (R.S., [Rya24])

$\mathsf{AC}_{\mathsf{WO}}$ *is equivalent to* $\mathrm{Suc} = \mathrm{Spec}_{\aleph}(M)$.

Failures of AC$_{\mathrm{WO}}$

Theorem (Karagila–R.S., [KR24])

There is a model M of ZF *such that*

$$\mathrm{Spec}_{\aleph}(M) = \{\langle \lambda, \kappa \rangle \mid \aleph_0 \leqslant \lambda \leqslant \kappa\}.$$

That is,

$$M \models (\forall \lambda \leqslant \kappa \textit{ infinite cardinals})(\exists X)\aleph(X) = \lambda \wedge \aleph^*(X) = \kappa.$$

Failures of AC_{WO}

Cohen's first model

Cohen's first model, M, is a model of ZF such that $L \subseteq M \subseteq L[(a_i)_{i<\omega}]$, where each a_i is a Cohen real. In fact, $M = L(A)$, where $A = \{a_i \mid i < \omega\}$.

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2. ZF proves that if $X \subseteq \mathbb{R}$ is infinite, then $|\aleph_0| \leqslant^* |X|$.
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Let α be an ordinal. An (indexed) family $X = \{X_\beta \mid \beta < \alpha\}$ of sets is α-*approachable* if for all $\beta < \alpha$: $X_\beta \neq \emptyset$ and $\{X_\gamma \mid \gamma < \beta\}$ has a choice function.

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EC_α was originally introduced as an unnamed axiom by Levy [Lév64][1] with the notation $C(\alpha)$. 'EC_α' is from [Rya25].

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