Local reflections of choice

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Forcing and AC

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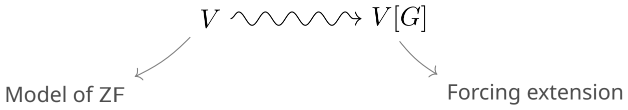

Forcing and AC

$$V \rightsquigarrow V[G]$$

Model of ZF

Forcing extension

Theorem

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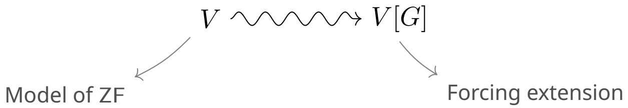

Theorem

1. *If* $V \models \mathsf{ZF}$ *then* $V[G] \models \mathsf{ZF}$.
2. *If* $V \models \mathsf{ZF} + \mathsf{AC}$ *then* $V[G] \models \mathsf{ZF} + \mathsf{AC}$.

Forcing and $\neg$AC

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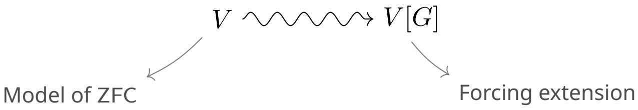
V

Model of ZFC

Forcing and ¬AC



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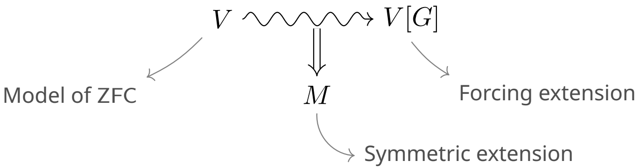

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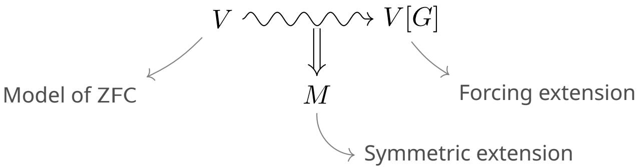

Let $\mathscr{G}$ act on $\mathbb{P}$. This action extends to $V^{\mathbb{P}}$ and we can restrict to a class HS of $\mathbb{P}$-names that are 'symmetric enough'. $M = \mathrm{HS}^G$.

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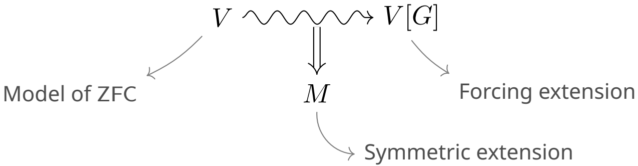

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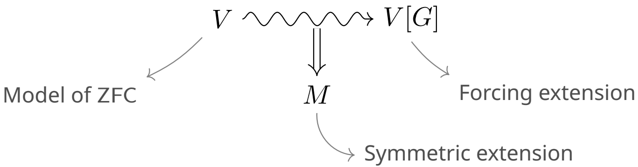

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1. *If* $V \models$ ZF *then* $M \models$ ZF.
2. *Usually* $M \models \neg$AC.

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Furthermore, SVC *holds if and only if there is a forcing* $\mathbb{P}$ *such that* $\mathbb{1}_{\mathbb{P}}\Vdash$ AC*.*

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4. M *is a symmetric extension of a model of* ZFC.

Local reflections of choice

SVC(S) means "for all X there is $\eta \in \mathrm{Ord}$ and $f\colon S \times \eta \twoheadrightarrow X$".

AC$_X$: If $\mathcal{A} = \{A_y \mid y \in X\} \not\ni \emptyset$ then $\mathcal{A}$ has a choice function.
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	Consequence of AC	Local reflection
(Blass)	AC	S can be well-ordered
(R.S.)	AC$_X$	AC$_X(S)$
(R.S.)	DC$_\lambda$	DC$_\lambda$ for subtrees of $S^{<\lambda}$
(Blass–Pincus)	BPI	There is a fine ultrafilter on $[S]^{<\omega}$
(Karagila–Schilhan)	KWP$_\alpha$	There is an injection $\mathscr{P}(S) \hookrightarrow \mathscr{P}^\alpha(\mathrm{Ord})$
(Karagila–Schilhan)	KWP$^*_\alpha$	There is a surjection $\mathscr{P}^\alpha(\mathrm{Ord}) \twoheadrightarrow S$.

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