

The Hartogs–Lindenbaum Spectrum

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The definition $|A| = \min\{\alpha \in \text{Ord} \mid |A| = |\alpha|\}$ is invalid.

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- ▶ $\aleph(X) \leq \aleph^*(X)$.

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^aFor all X , $\aleph^*(A \times \lambda) \leq \max\{\aleph^*(A)^+, \lambda^+\}$.

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Corollary

Either there are **no** eccentric sets, or there is a **proper class** of eccentric sets.

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Theorem (Pincus, R.S. [Pel78; RS24])

The following are equivalent:

1. For all X , $\aleph(X) = \aleph^*(X)$;
2. AC_{wo} ;
3. there is κ such that for all X , $\aleph^*(X) \geq \kappa \implies \aleph(X) = \aleph^*(X)$;
4. there is κ such that for all X , $\aleph(X) \geq \kappa \implies \aleph(X) = \aleph^*(X)$;
5. for all X , $\aleph(X)$ is regular; and
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When is either bound reached? When is either bound *possible* to reach?

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