The Hartogs–Lindenbaum Spectrum

Calliope Ryan-Smith

Univeristy of Leeds
Séminaire Général de Logique IMJ-PRG

30th September 2024

c.Ryan-Smith@leeds.ac.uk

https://academic.calliope.mx

How big is a set?

How big is a set?

Definition

Let X, Y be sets.

How big is a set?

Definition

Let X, Y be sets.

- $|X| \leqslant |Y|$ if there is an injection $X \to Y$.

How big is a set?

Definition

Let X, Y be sets.

- $|X| \leqslant |Y|$ if there is an injection $X \to Y$.
- $|X| \leqslant^* |Y|$ if there is a surjection $Y \to X$, or $X = \emptyset$.

How big is a set?

Definition

Let X, Y be sets.

- $|X| \leqslant |Y|$ if there is an injection $X \to Y$.
- $|X| \leqslant^* |Y|$ if there is a surjection $Y \to X$, or $X = \emptyset$.
- $|X| = |Y|$ if there is a bijection $X \to Y$.

How big is a set?

Definition

Let X, Y be sets.

- $|X| \leqslant |Y|$ if there is an injection $X \to Y$.
- $|X| \leqslant^* |Y|$ if there is a surjection $Y \to X$, or $X = \emptyset$.
- $|X| = |Y|$ if there is a bijection $X \to Y$.

Theorem (Zermelo, *Well-ordering principle*)

How big is a set?

Definition

Let X, Y be sets.

- $|X| \leqslant |Y|$ if there is an injection $X \to Y$.
- $|X| \leqslant^* |Y|$ if there is a surjection $Y \to X$, or $X = \emptyset$.
- $|X| = |Y|$ if there is a bijection $X \to Y$.

Theorem (Zermelo, *Well-ordering principle*)

(ZFC) For all X there is an ordinal α such that $|X| = |\alpha|$.

How big is a set?

Definition

Let X, Y be sets.

- $|X| \leqslant |Y|$ if there is an injection $X \to Y$.
- $|X| \leqslant^* |Y|$ if there is a surjection $Y \to X$, or $X = \emptyset$.
- $|X| = |Y|$ if there is a bijection $X \to Y$.

Theorem (Zermelo, *Well-ordering principle*)

(ZFC) For all X there is an ordinal α such that $|X| = |\alpha|$.

So the size of X is $|X| = \min\{\alpha \in \mathrm{Ord} \mid |X| = |\alpha|\}$.

(Some) consequences of the Well-ordering Principle

(Some) consequences of the Well-ordering Principle

1. $|X| \leqslant |Y|$ if and only if $|X| \leqslant^* |Y|$.

(Some) consequences of the Well-ordering Principle

1. $|X| \leqslant |Y|$ if and only if $|X| \leqslant^* |Y|$.
2. If X is infinite, then $|\omega| \leqslant |X|$ (and $|\omega| \leqslant^* |X|$).

(Some) consequences of the Well-ordering Principle

1. $|X| \leqslant |Y|$ if and only if $|X| \leqslant^* |Y|$.
2. If X is infinite, then $|\omega| \leqslant |X|$ (and $|\omega| \leqslant^* |X|$).
3. If X is infinite, then $|X \cup \{X\}| = |X|$.

(Some) consequences of the Well-ordering Principle

1. $|X| \leqslant |Y|$ if and only if $|X| \leqslant^* |Y|$.
2. If X is infinite, then $|\omega| \leqslant |X|$ (and $|\omega| \leqslant^* |X|$).
3. If X is infinite, then $|X \cup \{X\}| = |X|$.

Non-example (Cohen's first model)

(Some) consequences of the Well-ordering Principle

1. $|X| \leqslant |Y|$ if and only if $|X| \leqslant^* |Y|$.
2. If X is infinite, then $|\omega| \leqslant |X|$ (and $|\omega| \leqslant^* |X|$).
3. If X is infinite, then $|X \cup \{X\}| = |X|$.

Non-example (Cohen's first model)

In Cohen's first model of ZF $+ \neg$AC there is an infinite set A such that there is no injection $\omega \to A$. This violates Consequence (2)![a]

(Some) consequences of the Well-ordering Principle

1. $|X| \leqslant |Y|$ if and only if $|X| \leqslant^* |Y|$.
2. If X is infinite, then $|\omega| \leqslant |X|$ (and $|\omega| \leqslant^* |X|$).
3. If X is infinite, then $|X \cup \{X\}| = |X|$.

Non-example (Cohen's first model)

In Cohen's first model of ZF $+ \neg$AC there is an infinite set A such that there is no injection $\omega \to A$. This violates Consequence (2)![a] However, there *is* a surjection $A \to \omega$, also violating Consequence (1).

[a] This also violates Consequence (3) (exercise).

(Some) consequences of the Well-ordering Principle

1. $|X| \leqslant |Y|$ if and only if $|X| \leqslant^* |Y|$.
2. If X is infinite, then $|\omega| \leqslant |X|$ (and $|\omega| \leqslant^* |X|$).
3. If X is infinite, then $|X \cup \{X\}| = |X|$.

Non-example (Cohen's first model)

In Cohen's first model of ZF $+ \neg$AC there is an infinite set A such that there is no injection $\omega \to A$. This violates Consequence (2)![a] However, there *is* a surjection $A \to \omega$, also violating Consequence (1).

[a] This also violates Consequence (3) (exercise).

The definition $|A| = \min\{\alpha \in \mathrm{Ord} \mid |A| = |\alpha|\}$ is invalid.

Approximate size

Approximate size

Definition (Hartogs and Lindenbaum numbers)

Let X be a set.

Approximate size

Definition (Hartogs and Lindenbaum numbers)

Let X be a set. The **Hartogs number** of X, denoted $\aleph(X)$, is

$$\min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq |X|\}.$$

Approximate size

Definition (Hartogs and Lindenbaum numbers)

Let X be a set. The **Hartogs number** of X, denoted $\aleph(X)$, is

$$\min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq |X|\}.$$

The **Lindenbaum number** of X, denoted $\aleph^*(X)$, is

$$\min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq^* |X|\}.$$

Approximate size

Definition (Hartogs and Lindenbaum numbers)

Let X be a set. The **Hartogs number** of X, denoted $\aleph(X)$, is

$$\min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq |X|\}.$$

The **Lindenbaum number** of X, denoted $\aleph^*(X)$, is

$$\min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq^* |X|\}.$$

Fact/Theorem (Hartogs/Lindenbaum [Har15; LT26])/Exercise

For all X:

Approximate size

Definition (Hartogs and Lindenbaum numbers)

Let X be a set. The **Hartogs number** of X, denoted $\aleph(X)$, is

$$\min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq |X|\}.$$

The **Lindenbaum number** of X, denoted $\aleph^*(X)$, is

$$\min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq^* |X|\}.$$

Fact/Theorem (Hartogs/Lindenbaum [Har15; LT26])/Exercise

For all X:

- $\aleph(X)$ and $\aleph^*(X)$ exist.

Approximate size

Definition (Hartogs and Lindenbaum numbers)

Let X be a set. The **Hartogs number** of X, denoted $\aleph(X)$, is

$$\min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq |X|\}.$$

The **Lindenbaum number** of X, denoted $\aleph^*(X)$, is

$$\min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq^* |X|\}.$$

Fact/Theorem (Hartogs/Lindenbaum [Har15; LT26])/Exercise

For all X:

- $\aleph(X)$ and $\aleph^*(X)$ exist.
- $\aleph(X)$ and $\aleph^*(X)$ are cardinals (i.e. $|\alpha| < |\aleph(X)|$ for all $\alpha < \aleph(X)$).

Approximate size

Definition (Hartogs and Lindenbaum numbers)

Let X be a set. The **Hartogs number** of X, denoted $\aleph(X)$, is

$$\min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant |X|\}.$$

The **Lindenbaum number** of X, denoted $\aleph^*(X)$, is

$$\min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant^* |X|\}.$$

Fact/Theorem (Hartogs/Lindenbaum [Har15; LT26])/Exercise

For all X:

- $\aleph(X)$ and $\aleph^*(X)$ exist.
- $\aleph(X)$ and $\aleph^*(X)$ are cardinals (i.e. $|\alpha| < |\aleph(X)|$ for all $\alpha < \aleph(X)$).
- $\aleph(X) \leqslant \aleph^*(X)$.

$$\aleph(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq |X|\}, \qquad \aleph^*(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq^* |X|\}$$

$$\aleph(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq |X|\}, \qquad \aleph^*(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq^* |X|\}$$

Example

If X can be well-ordered, say $|X| = \lambda$, then $\aleph(X) = \aleph^*(X) = \lambda^+$.

$$\aleph(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq |X|\}, \qquad \aleph^*(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq^* |X|\}$$

Example

If X can be well-ordered, say $|X| = \lambda$, then $\aleph(X) = \aleph^*(X) = \lambda^+$.

In Cohen's first model, $\aleph(A) = \omega$, and $\aleph^*(A) = \omega_1$.

$$\aleph(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq |X|\}, \qquad \aleph^*(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq^* |X|\}$$

Example

If X can be well-ordered, say $|X| = \lambda$, then $\aleph(X) = \aleph^*(X) = \lambda^+$.

In Cohen's first model, $\aleph(A) = \omega$, and $\aleph^*(A) = \omega_1$.

Definition

A set X is **eccentric** if $\aleph(X) < \aleph^*(X)$.

$$\aleph(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq |X|\}, \qquad \aleph^*(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq^* |X|\}$$

Example

If X can be well-ordered, say $|X| = \lambda$, then $\aleph(X) = \aleph^*(X) = \lambda^+$.
In Cohen's first model, $\aleph(A) = \omega$, and $\aleph^*(A) = \omega_1$.

Definition

A set X is **eccentric** if $\aleph(X) < \aleph^*(X)$.

So if $V \vDash$ ZFC then there are *no* eccentric sets:

$\aleph(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq |X|\}, \qquad \aleph^*(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq^* |X|\}$

Example

If X can be well-ordered, say $|X| = \lambda$, then $\aleph(X) = \aleph^*(X) = \lambda^+$.

In Cohen's first model, $\aleph(A) = \omega$, and $\aleph^*(A) = \omega_1$.

Definition

A set X is **eccentric** if $\aleph(X) < \aleph^*(X)$.

So if $V \vDash \mathrm{ZFC}$ then there are *no* eccentric sets: $\aleph(X) = \aleph^*(X) = |X|^+$.

$$\aleph(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq |X|\}, \qquad \aleph^*(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq^* |X|\}$$

Example

If X can be well-ordered, say $|X| = \lambda$, then $\aleph(X) = \aleph^*(X) = \lambda^+$.
In Cohen's first model, $\aleph(A) = \omega$, and $\aleph^*(A) = \omega_1$.

Definition

A set X is **eccentric** if $\aleph(X) < \aleph^*(X)$.

So if $V \vDash \mathrm{ZFC}$ then there are *no* eccentric sets: $\aleph(X) = \aleph^*(X) = |X|^+$.

Question

What are the eccentric sets in Cohen's first model?

Question

What are the eccentric sets in Cohen's first model?

Question

What are the eccentric sets in Cohen's first model?

Theory. Maybe all eccentric sets look like A: If X is eccentric then $\aleph(X) = \omega$ and $\aleph^*(X) = \omega_1$.

Question

What are the eccentric sets in Cohen's first model?

Theory. Maybe all eccentric sets look like A: If X is eccentric then $\aleph(X) = \omega$ and $\aleph^*(X) = \omega_1$.

Answer. No.

Question

What are the eccentric sets in Cohen's first model?

Theory. Maybe all eccentric sets look like A: If X is eccentric then $\aleph(X) = \omega$ and $\aleph^*(X) = \omega_1$.

Answer. No.

$\aleph^*(A) = \omega_1$, so there is a partition $A = \bigsqcup_{n \in \omega} A_n$ such that $A_n \neq \emptyset$ for all n.

Question

What are the eccentric sets in Cohen's first model?

Theory. Maybe all eccentric sets look like A: If X is eccentric then $\aleph(X) = \omega$ and $\aleph^*(X) = \omega_1$.

Answer. No.

$\aleph^*(A) = \omega_1$, so there is a partition $A = \bigsqcup_{n \in \omega} A_n$ such that $A_n \neq \emptyset$ for all n. Let

$$B_m = \bigcup_{n<m} A_n \times \omega_n, \qquad B = \bigcup_{n \in \omega} A_n \times \omega_n.$$

Question

What are the eccentric sets in Cohen's first model?

Theory. Maybe all eccentric sets look like A: If X is eccentric then $\aleph(X) = \omega$ and $\aleph^*(X) = \omega_1$.

Answer. No.

$\aleph^*(A) = \omega_1$, so there is a partition $A = \bigsqcup_{n\in\omega} A_n$ such that $A_n \neq \emptyset$ for all n. Let

$$B_m = \bigcup_{n<m} A_n \times \omega_n, \qquad B = \bigcup_{n\in\omega} A_n \times \omega_n.$$

Lemma

$\aleph(B) = \omega_\omega$ *and* $\aleph^*(B) = \omega_{\omega+1}$.

$$B_m = \bigcup_{n<m} A_n \times \omega_n, \qquad B = \bigcup_{n\in\omega} A_n \times \omega_n.$$

Lemma

$\aleph(B) = \omega_\omega$ and $\aleph^*(B) = \omega_{\omega+1}$.

$$B_m = \bigcup_{n<m} A_n \times \omega_n, \qquad B = \bigcup_{n\in\omega} A_n \times \omega_n.$$

Lemma

$\aleph(B) = \omega_\omega$ *and* $\aleph^*(B) = \omega_{\omega+1}$.

Proof.

($\aleph(B) = \omega_\omega$).

$$B_m = \bigcup_{n<m} A_n \times \omega_n, \qquad B = \bigcup_{n\in\omega} A_n \times \omega_n.$$

Lemma

$\aleph(B) = \omega_\omega$ *and* $\aleph^*(B) = \omega_{\omega+1}$.

Proof.

($\aleph(B) = \omega_\omega$). For all $n \in \omega$, $\omega_n \hookrightarrow A_n \times \omega_n \hookrightarrow B$,

$$B_m = \bigcup_{n<m} A_n \times \omega_n, \qquad B = \bigcup_{n\in\omega} A_n \times \omega_n.$$

Lemma

$\aleph(B) = \omega_\omega$ *and* $\aleph^*(B) = \omega_{\omega+1}$.

Proof.

($\aleph(B) = \omega_\omega$). For all $n \in \omega$, $\omega_n \hookrightarrow A_n \times \omega_n \hookrightarrow B$, so $\aleph(B) \geqslant \omega_\omega$.

$$B_m = \bigcup_{n<m} A_n \times \omega_n, \qquad B = \bigcup_{n\in\omega} A_n \times \omega_n.$$

Lemma

$\aleph(B) = \omega_\omega$ *and* $\aleph^*(B) = \omega_{\omega+1}$.

Proof.

($\aleph(B) = \omega_\omega$). For all $n \in \omega$, $\omega_n \hookrightarrow A_n \times \omega_n \hookrightarrow B$, so $\aleph(B) \geqslant \omega_\omega$.

Fact/exercise: For all $m \in \omega$, $\aleph(B_m) = \omega_m$.

$$B_m = \bigcup_{n<m} A_n \times \omega_n, \qquad B = \bigcup_{n\in\omega} A_n \times \omega_n.$$

Lemma

$\aleph(B) = \omega_\omega$ *and* $\aleph^*(B) = \omega_{\omega+1}$.

Proof.

($\aleph(B) = \omega_\omega$). For all $n \in \omega$, $\omega_n \hookrightarrow A_n \times \omega_n \hookrightarrow B$, so $\aleph(B) \geqslant \omega_\omega$.

Fact/exercise: For all $m \in \omega$, $\aleph(B_m) = \omega_m$.

So if $f : \omega_\omega \hookrightarrow B$, then $f``\omega_\omega \not\subseteq B_m$ for all m.

$$B_m = \bigcup_{n<m} A_n \times \omega_n, \qquad B = \bigcup_{n\in\omega} A_n \times \omega_n.$$

Lemma

$\aleph(B) = \omega_\omega$ *and* $\aleph^*(B) = \omega_{\omega+1}$.

Proof.

($\aleph(B) = \omega_\omega$). For all $n \in \omega$, $\omega_n \hookrightarrow A_n \times \omega_n \hookrightarrow B$, so $\aleph(B) \geqslant \omega_\omega$.

Fact/exercise: For all $m \in \omega$, $\aleph(B_m) = \omega_m$.

So if $f\colon \omega_\omega \hookrightarrow B$, then $f``\omega_\omega \not\subseteq B_m$ for all m. Then f lets us define an injection $\omega \to A$, a contradiction.

$$B_m = \bigcup_{n<m} A_n \times \omega_n, \qquad B = \bigcup_{n\in\omega} A_n \times \omega_n.$$

Lemma

$\aleph(B) = \omega_\omega$ *and* $\aleph^*(B) = \omega_{\omega+1}$.

Proof.

($\aleph(B) = \omega_\omega$). For all $n \in \omega$, $\omega_n \hookrightarrow A_n \times \omega_n \hookrightarrow B$, so $\aleph(B) \geqslant \omega_\omega$.

Fact/exercise: For all $m \in \omega$, $\aleph(B_m) = \omega_m$.

So if $f : \omega_\omega \hookrightarrow B$, then $f``\omega_\omega \not\subseteq B_m$ for all m. Then f lets us define an injection $\omega \to A$, a contradiction.

($\aleph^*(B) = \omega_{\omega+1}$).

$$B_m = \bigcup_{n<m} A_n \times \omega_n, \qquad B = \bigcup_{n\in\omega} A_n \times \omega_n.$$

Lemma

$\aleph(B) = \omega_\omega$ *and* $\aleph^*(B) = \omega_{\omega+1}$.

Proof.

($\aleph(B) = \omega_\omega$). For all $n \in \omega$, $\omega_n \hookrightarrow A_n \times \omega_n \hookrightarrow B$, so $\aleph(B) \geqslant \omega_\omega$.

Fact/exercise: For all $m \in \omega$, $\aleph(B_m) = \omega_m$.

So if $f : \omega_\omega \hookrightarrow B$, then $f``\omega_\omega \not\subseteq B_m$ for all m. Then f lets us define an injection $\omega \to A$, a contradiction.

($\aleph^*(B) = \omega_{\omega+1}$). Projection is a surjection $B \to \omega_\omega$,

$$B_m = \bigcup_{n<m} A_n \times \omega_n, \qquad B = \bigcup_{n\in\omega} A_n \times \omega_n.$$

Lemma

$\aleph(B) = \omega_\omega$ *and* $\aleph^*(B) = \omega_{\omega+1}$.

Proof.

($\aleph(B) = \omega_\omega$). For all $n \in \omega$, $\omega_n \hookrightarrow A_n \times \omega_n \hookrightarrow B$, so $\aleph(B) \geqslant \omega_\omega$.

Fact/exercise: For all $m \in \omega$, $\aleph(B_m) = \omega_m$.

So if $f : \omega_\omega \hookrightarrow B$, then $f``\omega_\omega \not\subseteq B_m$ for all m. Then f lets us define an injection $\omega \to A$, a contradiction.

($\aleph^*(B) = \omega_{\omega+1}$). Projection is a surjection $B \to \omega_\omega$, so $\aleph^*(B) \geqslant \omega_{\omega+1}$.

$$B_m = \bigcup_{n<m} A_n \times \omega_n, \qquad B = \bigcup_{n\in\omega} A_n \times \omega_n.$$

Lemma

$\aleph(B) = \omega_\omega$ *and* $\aleph^*(B) = \omega_{\omega+1}$.

Proof.

($\aleph(B) = \omega_\omega$). For all $n \in \omega$, $\omega_n \hookrightarrow A_n \times \omega_n \hookrightarrow B$, so $\aleph(B) \geqslant \omega_\omega$.

Fact/exercise: For all $m \in \omega$, $\aleph(B_m) = \omega_m$.

So if $f : \omega_\omega \hookrightarrow B$, then $f``\omega_\omega \not\subseteq B_m$ for all m. Then f lets us define an injection $\omega \to A$, a contradiction.

($\aleph^*(B) = \omega_{\omega+1}$). Projection is a surjection $B \to \omega_\omega$, so $\aleph^*(B) \geqslant \omega_{\omega+1}$.

$B \subseteq A \times \omega_\omega$,

$$B_m = \bigcup_{n<m} A_n \times \omega_n, \qquad B = \bigcup_{n\in\omega} A_n \times \omega_n.$$

Lemma

$\aleph(B) = \omega_\omega$ *and* $\aleph^*(B) = \omega_{\omega+1}$.

Proof.

($\aleph(B) = \omega_\omega$). For all $n \in \omega$, $\omega_n \hookrightarrow A_n \times \omega_n \hookrightarrow B$, so $\aleph(B) \geqslant \omega_\omega$.

Fact/exercise: For all $m \in \omega$, $\aleph(B_m) = \omega_m$.

So if $f : \omega_\omega \hookrightarrow B$, then $f``\omega_\omega \nsubseteq B_m$ for all m. Then f lets us define an injection $\omega \to A$, a contradiction.

($\aleph^*(B) = \omega_{\omega+1}$). Projection is a surjection $B \to \omega_\omega$, so $\aleph^*(B) \geqslant \omega_{\omega+1}$.

$B \subseteq A \times \omega_\omega$, so $\aleph^*(B) \leqslant \aleph^*(A \times \omega_\omega) \leqslant \aleph^*(A)^+ \times \omega_{\omega+1} = \omega_{\omega+1}$.[a] □

[a] For all X, $\aleph^*(A \times \lambda) \leqslant \max\{\aleph^*(A)^+, \lambda^+\}$.

Looking at the proof, if $\mathrm{cf}(\lambda) = \omega$

Looking at the proof, if $\mathrm{cf}(\lambda) = \omega$ then the same construction works with ω_ω replaced by λ.

Looking at the proof, if $\mathrm{cf}(\lambda) = \omega$ then the same construction works with ω_ω replaced by λ.

Lemma (R.S. [RS24])

(Cohen's first model)

Looking at the proof, if $\mathrm{cf}(\lambda) = \omega$ then the same construction works with ω_ω replaced by λ.

Lemma (R.S. [RS24])

(Cohen's first model) If $\mathrm{cf}(\lambda) = \omega$ *then there is* X *such that* $\aleph(X) = \lambda$ *and* $\aleph^*(X) = \lambda^+$.

Looking at the proof, if $\mathrm{cf}(\lambda) = \omega$ then the same construction works with ω_ω replaced by λ.

Lemma (R.S. [RS24])

(Cohen's first model) If $\mathrm{cf}(\lambda) = \omega$ *then there is* X *such that* $\aleph(X) = \lambda$ *and* $\aleph^*(X) = \lambda^+$*.*

This is *all* of the eccentric sets.

Looking at the proof, if $\mathrm{cf}(\lambda) = \omega$ then the same construction works with ω_ω replaced by λ.

Lemma (R.S. [RS24])

(Cohen's first model) If $\mathrm{cf}(\lambda) = \omega$ *then there is* X *such that* $\aleph(X) = \lambda$ *and* $\aleph^*(X) = \lambda^+$.

This is *all* of the eccentric sets.

Fact

(Cohen's first model). For all X*, either:*

Looking at the proof, if $\mathrm{cf}(\lambda) = \omega$ then the same construction works with ω_ω replaced by λ.

Lemma (R.S. [RS24])

(Cohen's first model) If $\mathrm{cf}(\lambda) = \omega$ then there is X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \lambda^+$.

This is *all* of the eccentric sets.

Fact

(Cohen's first model). For all X, either:

- $\aleph(X) = \aleph^*(X) = \kappa^+$ *some* κ*, or*

Looking at the proof, if $\mathrm{cf}(\lambda) = \omega$ then the same construction works with ω_ω replaced by λ.

Lemma (R.S. [RS24])

(Cohen's first model) If $\mathrm{cf}(\lambda) = \omega$ then there is X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \lambda^+$.

This is *all* of the eccentric sets.

Fact

(Cohen's first model). For all X, either:

- *$\aleph(X) = \aleph^*(X) = \kappa^+$ some κ, or*
- *$\aleph(X) = \lambda$, $\aleph^*(X) = \lambda^+$, and $\mathrm{cf}(\lambda) = \omega$.*

Definition (Hartogs–Lindenbaum spectrum)

Definition (Hartogs–Lindenbaum spectrum)

Let $M \vDash \mathsf{ZF}$.

Definition (Hartogs–Lindenbaum spectrum)

Let $M \vDash \mathrm{ZF}$. The **Hartogs–Lindenbaum spectrum** of M is

Definition (Hartogs–Lindenbaum spectrum)

Let $M \vDash \mathrm{ZF}$. The **Hartogs–Lindenbaum spectrum** of M is

$$\mathrm{Spec}_{\aleph}(M) := \{\langle \aleph(X), \aleph^*(X) \rangle \mid X \in M\},$$

Definition (Hartogs–Lindenbaum spectrum)

Let $M \vDash \mathsf{ZF}$. The **Hartogs–Lindenbaum spectrum** of M is

$$\mathrm{Spec}_{\aleph}(M) := \{\langle \aleph(X), \aleph^*(X)\rangle \mid X \in M\},$$

the class of pairs $\langle \lambda, \kappa \rangle$ such that

Definition (Hartogs–Lindenbaum spectrum)

Let $M \vDash \mathsf{ZF}$. The **Hartogs–Lindenbaum spectrum** of M is

$$\mathrm{Spec}_{\aleph}(M) := \{\langle \aleph(X), \aleph^*(X)\rangle \mid X \in M\},$$

the class of pairs $\langle \lambda, \kappa\rangle$ such that $(\exists X \in M) M \vDash \aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$.

Definition (Hartogs–Lindenbaum spectrum)

Let $M \vDash \mathrm{ZF}$. The **Hartogs–Lindenbaum spectrum** of M is

$$\mathrm{Spec}_{\aleph}(M) := \{\langle \aleph(X), \aleph^*(X)\rangle \mid X \in M\},$$

the class of pairs $\langle \lambda, \kappa\rangle$ such that $(\exists X \in M) M \vDash \aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$.

For all well-ordered cardinals λ, $\aleph(\lambda) = \aleph^*(\lambda) = \lambda^+$,

Definition (Hartogs–Lindenbaum spectrum)

Let $M \vDash \text{ZF}$. The **Hartogs–Lindenbaum spectrum** of M is

$$\text{Spec}_{\aleph}(M) := \{\langle \aleph(X), \aleph^*(X) \rangle \mid X \in M\},$$

the class of pairs $\langle \lambda, \kappa \rangle$ such that $(\exists X \in M) M \vDash \aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$.

For all well-ordered cardinals λ, $\aleph(\lambda) = \aleph^*(\lambda) = \lambda^+$, so for all $M \vDash \text{ZF}$,

Definition (Hartogs–Lindenbaum spectrum)

Let $M \vDash \mathrm{ZF}$. The **Hartogs–Lindenbaum spectrum** of M is

$$\mathrm{Spec}_{\aleph}(M) := \{\langle \aleph(X), \aleph^*(X)\rangle \mid X \in M\},$$

the class of pairs $\langle \lambda, \kappa\rangle$ such that $(\exists X \in M) M \vDash \aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$.

For all well-ordered cardinals λ, $\aleph(\lambda) = \aleph^*(\lambda) = \lambda^+$, so for all $M \vDash \mathrm{ZF}$,

$$\mathrm{Suc} := \left\{\langle \lambda^+, \lambda^+\rangle \;\middle|\; \lambda \text{ a cardinal}\right\} \subseteq \mathrm{Spec}_{\aleph}(M).$$

Example

Definition (Hartogs–Lindenbaum spectrum)

Let $M \vDash$ ZF. The **Hartogs–Lindenbaum spectrum** of M is

$$\mathrm{Spec}_{\aleph}(M) := \{\langle \aleph(X), \aleph^*(X)\rangle \mid X \in M\},$$

the class of pairs $\langle \lambda, \kappa\rangle$ such that $(\exists X \in M) M \vDash \aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$.

For all well-ordered cardinals λ, $\aleph(\lambda) = \aleph^*(\lambda) = \lambda^+$, so for all $M \vDash$ ZF,

$$\mathrm{Suc} := \left\{\langle \lambda^+, \lambda^+\rangle \;\middle|\; \lambda \text{ a cardinal}\right\} \subseteq \mathrm{Spec}_{\aleph}(M).$$

Example

- If $M \vDash$ ZFC then $\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc}$.

Definition (Hartogs–Lindenbaum spectrum)

Let $M \vDash$ ZF. The **Hartogs–Lindenbaum spectrum** of M is

$$\mathrm{Spec}_{\aleph}(M) := \{\langle \aleph(X), \aleph^*(X)\rangle \mid X \in M\},$$

the class of pairs $\langle \lambda, \kappa\rangle$ such that $(\exists X \in M) M \vDash \aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$.

For all well-ordered cardinals λ, $\aleph(\lambda) = \aleph^*(\lambda) = \lambda^+$, so for all $M \vDash$ ZF,

$$\mathrm{Suc} := \left\{\langle \lambda^+, \lambda^+\rangle \;\middle|\; \lambda \text{ a cardinal}\right\} \subseteq \mathrm{Spec}_{\aleph}(M).$$

Example

- If $M \vDash$ ZFC then $\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc}$.
- If M is Cohen's first model, $\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc} \cup \{\langle \lambda, \lambda^+\rangle \mid \mathrm{cf}(\lambda) = \omega\}$.

Definition (Hartogs–Lindenbaum spectrum)

Let $M \vDash \mathsf{ZF}$. The **Hartogs–Lindenbaum spectrum** of M is

$$\mathrm{Spec}_{\aleph}(M) := \{\langle \aleph(X), \aleph^*(X)\rangle \mid X \in M\},$$

the class of pairs $\langle \lambda, \kappa\rangle$ such that $(\exists X \in M) M \vDash \aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$.

For all well-ordered cardinals λ, $\aleph(\lambda) = \aleph^*(\lambda) = \lambda^+$, so for all $M \vDash \mathsf{ZF}$,

$$\mathrm{Suc} := \left\{\langle \lambda^+, \lambda^+\rangle \;\middle|\; \lambda \text{ a cardinal}\right\} \subseteq \mathrm{Spec}_{\aleph}(M).$$

Example

- If $M \vDash \mathsf{ZFC}$ then $\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc}$.
- If M is Cohen's first model, $\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc} \cup \{\langle \lambda, \lambda^+\rangle \mid \mathrm{cf}(\lambda) = \omega\}$.
- [KRS23] There is $M \vDash \mathsf{ZF}$ such that $\mathrm{Spec}_{\aleph}(M) = \{\langle n+1, n+1\rangle \mid n \in \omega\} \cup \{\langle \lambda, \kappa\rangle \mid \lambda \leqslant \kappa \text{ are infinite}\}$.

Theorem (R.S. [RS24])

Theorem (R.S. [RS24])

If there is A such that $\aleph(A) \leqslant \mu < \aleph^*(A)$, where μ is a regular cardinal,

Theorem (R.S. [RS24])

If there is A *such that* $\aleph(A) \leqslant \mu < \aleph^*(A)$*, where* μ *is a regular cardinal, then for all* λ *large enough such that* $\mathrm{cf}(\lambda) = \mu$*,*

Theorem (R.S. [RS24])

If there is A such that $\aleph(A) \leqslant \mu < \aleph^*(A)$, where μ is a regular cardinal, then for all λ large enough such that $\mathrm{cf}(\lambda) = \mu$, there is a set B such that $\aleph(B) = \lambda$ and $\aleph^*(B) = \lambda^+$.

Theorem (R.S. [RS24])

If there is A such that $\aleph(A) \leqslant \mu < \aleph^*(A)$, where μ is a regular cardinal, then for all λ large enough such that $\mathrm{cf}(\lambda) = \mu$, there is a set B such that $\aleph(B) = \lambda$ and $\aleph^*(B) = \lambda^+$.

The proof is the same as in Cohen's first model:

Theorem (R.S. [RS24])

If there is A such that $\aleph(A) \leqslant \mu < \aleph^*(A)$, where μ is a regular cardinal, then for all λ large enough such that $\mathrm{cf}(\lambda) = \mu$, there is a set B such that $\aleph(B) = \lambda$ and $\aleph^*(B) = \lambda^+$.

The proof is the same as in Cohen's first model: If $A = \bigsqcup_{\alpha \in \mu} A_\alpha$

Theorem (R.S. [RS24])

If there is A such that $\aleph(A) \leqslant \mu < \aleph^*(A)$, where μ is a regular cardinal, then for all λ large enough such that $\mathrm{cf}(\lambda) = \mu$, there is a set B such that $\aleph(B) = \lambda$ and $\aleph^*(B) = \lambda^+$.

The proof is the same as in Cohen's first model: If $A = \bigsqcup_{\alpha \in \mu} A_\alpha$ and $\lambda = \sup\{\lambda_\alpha \mid \alpha \in \mu\}$,

Theorem (R.S. [RS24])

If there is A such that $\aleph(A) \leqslant \mu < \aleph^*(A)$, where μ is a regular cardinal, then for all λ large enough such that $\mathrm{cf}(\lambda) = \mu$, there is a set B such that $\aleph(B) = \lambda$ and $\aleph^*(B) = \lambda^+$.

The proof is the same as in Cohen's first model: If $A = \bigsqcup_{\alpha\in\mu} A_\alpha$ and $\lambda = \sup\{\lambda_\alpha \mid \alpha \in \mu\}$, then let $B = \bigcup_{\alpha\in\mu} A_\alpha \times \lambda_\alpha$.

Theorem (R.S. [RS24])

If there is A such that $\aleph(A) \leqslant \mu < \aleph^*(A)$, where μ is a regular cardinal, then for all λ large enough such that $\mathrm{cf}(\lambda) = \mu$, there is a set B such that $\aleph(B) = \lambda$ and $\aleph^*(B) = \lambda^+$.

The proof is the same as in Cohen's first model: If $A = \bigsqcup_{\alpha \in \mu} A_\alpha$ and $\lambda = \sup\{\lambda_\alpha \mid \alpha \in \mu\}$, then let $B = \bigcup_{\alpha \in \mu} A_\alpha \times \lambda_\alpha$.

Corollary

*Either there are **no** eccentric sets, or there is a **proper class** of eccentric sets.*

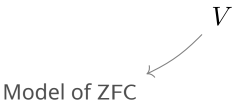
Symmetric extensions

Symmetric extensions

Instead of considering *all* models of ZF, let us consider only *symmetric extensions*:

Symmetric extensions

Instead of considering *all* models of ZF, let us consider only *symmetric extensions*:



Symmetric extensions

Instead of considering *all* models of ZF, let us consider only *symmetric extensions*:

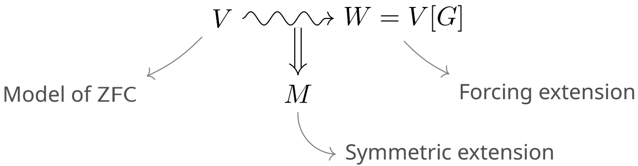
$$V \rightsquigarrow W = V[G]$$

Model of ZFC

Forcing extension

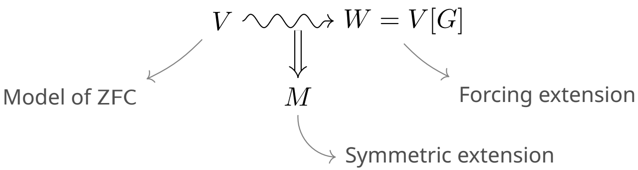
Symmetric extensions

Instead of considering *all* models of ZF, let us consider only *symmetric extensions*:



Symmetric extensions

Instead of considering *all* models of ZF, let us consider only *symmetric extensions*:



Many, many models of ZF $+ \neg$AC are obtained as symmetric extensions.

SVC

SVC

Definition

For $M \vDash \mathsf{ZF}$ and $A \in M$,

SVC

Definition

For $M \vDash$ ZF and $A \in M$, SVC(A) is the statement

SVC

Definition

For $M \vDash$ ZF and $A \in M$, SVC(A) is the statement

$$(\forall X)(\exists \eta \in \mathrm{Ord})\, |X| \leqslant^* |A \times \eta| \,.$$

SVC

Definition

For $M \vDash$ ZF and $A \in M$, SVC(A) is the statement

$$(\forall X)(\exists \eta \in \mathrm{Ord})\, |X| \leqslant^* |A \times \eta|\,.$$

That is, for all $X \in M$

SVC

Definition

For $M \vDash$ ZF and $A \in M$, SVC(A) is the statement

$$(\forall X)(\exists \eta \in \mathrm{Ord}) \, |X| \leqslant^* |A \times \eta| \, .$$

That is, for all $X \in M$ there is an ordinal η

SVC

Definition

For $M \vDash$ ZF and $A \in M$, SVC(A) is the statement

$$(\forall X)(\exists \eta \in \mathrm{Ord})\, |X| \leqslant^* |A \times \eta|\,.$$

That is, for all $X \in M$ there is an ordinal η such that there is a surjection $f \colon A \times \eta \to X$ in M.

SVC

Definition

For $M \vDash$ ZF and $A \in M$, SVC(A) is the statement

$$(\forall X)(\exists \eta \in \mathrm{Ord}) \, |X| \leqslant^* |A \times \eta| \, .$$

That is, for all $X \in M$ there is an ordinal η such that there is a surjection $f \colon A \times \eta \to X$ in M.
We write SVC, **small violations of choice**, to mean $(\exists A)$SVC(A).

SVC

Definition

For $M \vDash$ ZF and $A \in M$, SVC(A) is the statement

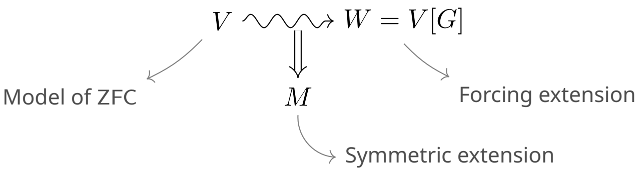
$$(\forall X)(\exists \eta \in \mathrm{Ord})\, |X| \leqslant^* |A \times \eta|\,.$$

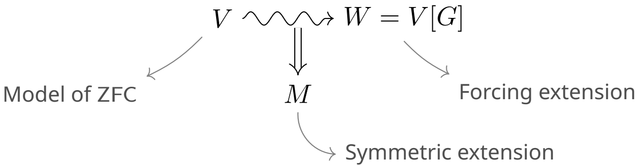
That is, for all $X \in M$ there is an ordinal η such that there is a surjection $f\colon A \times \eta \to X$ in M.

We write SVC, **small violations of choice**, to mean $(\exists A)$SVC(A).

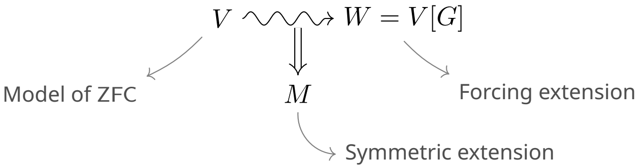
Theorem (Blass/Usuba [Bla79; Usu21])

M is a symmetric extension of a model of ZFC *if and only if* $M \vDash$ SVC.


V
W = V[G]
M
Model of ZFC
Forcing extension
Symmetric extension



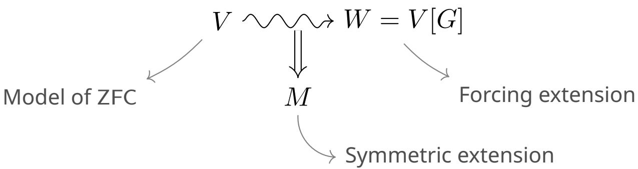
We will assume that M and W agree on cardinality and cofinality of ordinals.



We will assume that M and W agree on cardinality and cofinality of ordinals.

Example

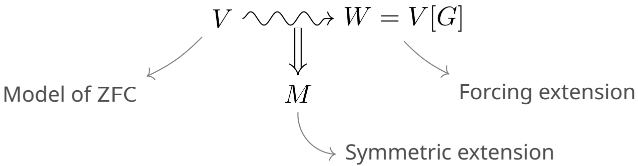
Cohen's first model is a model of SVC(A), where A is our eccentric set from earlier.



We will assume that M and W agree on cardinality and cofinality of ordinals.

Example

Cohen's first model is a model of SVC(A), where A is our eccentric set from earlier. Also, $W \vDash |A| = \omega$.

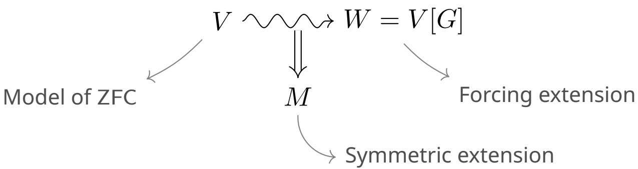


We will assume that M and W agree on cardinality and cofinality of ordinals.

Example

Cohen's first model is a model of SVC(A), where A is our eccentric set from earlier. Also, $W \vDash |A| = \omega$.[a]

[a] This is why $\aleph^*(A) < \omega_2$!

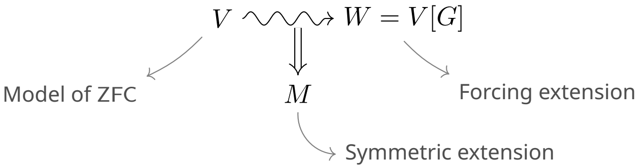


We will assume that M and W agree on cardinality and cofinality of ordinals.

Example

Cohen's first model is a model of SVC(A), where A is our eccentric set from earlier. Also, $W \vDash |A| = \omega$.[a]

[a]This is why $\aleph^*(A) < \omega_2$! If $M \vDash |\omega_1| \leqslant^* |A|$,



We will assume that M and W agree on cardinality and cofinality of ordinals.

Example

Cohen's first model is a model of SVC(A), where A is our eccentric set from earlier. Also, $W \vDash |A| = \omega$.[a]

[a] This is why $\aleph^*(A) < \omega_2$! If $M \vDash |\omega_1| \leqslant^* |A|$, then $W \vDash |\omega_1| \leqslant^* |A| = |\omega|$.

Theorem (R.S. [RS24])

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$*,*

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$, *then* $\mathrm{cf}(\mu) \leqslant |A|^W$.

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$, *then* $\mathrm{cf}(\mu) \leqslant |A|^W$.

Proof.

Let $\eta \in \mathrm{Ord}$ be such that $|X| \leqslant^* |A \times \eta|$, say $g \colon A \times \eta \twoheadrightarrow X$.

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$, *then* $\mathrm{cf}(\mu) \leqslant |A|^W$.

Proof.

Let $\eta \in \mathrm{Ord}$ be such that $|X| \leqslant^* |A \times \eta|$, say $g\colon A \times \eta \twoheadrightarrow X$.

For $a \in A$, let $X_a = g``(\{a\} \times \eta)$.

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$*, then* $\mathrm{cf}(\mu) \leqslant |A|^W$*.*

Proof.

Let $\eta \in \mathrm{Ord}$ be such that $|X| \leqslant^* |A \times \eta|$, say $g\colon A \times \eta \twoheadrightarrow X$.

For $a \in A$, let $X_a = g``(\{a\} \times \eta)$. Note that X_a is well-orderable in M.

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$*, then* $\mathrm{cf}(\mu) \leqslant |A|^W$*.*

Proof.

Let $\eta \in \mathrm{Ord}$ be such that $|X| \leqslant^* |A \times \eta|$, say $g\colon A \times \eta \twoheadrightarrow X$.

For $a \in A$, let $X_a = g``(\{a\} \times \eta)$. Note that X_a is well-orderable in M.

Let $\mu < \aleph^*(X)$, so there is a surjection $f\colon X \twoheadrightarrow \mu$.

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$, *then* $\operatorname{cf}(\mu) \leqslant |A|^W$.

Proof.

Let $\eta \in \mathrm{Ord}$ be such that $|X| \leqslant^* |A \times \eta|$, say $g\colon A \times \eta \twoheadrightarrow X$.

For $a \in A$, let $X_a = g``(\{a\} \times \eta)$. Note that X_a is well-orderable in M.

Let $\mu < \aleph^*(X)$, so there is a surjection $f\colon X \twoheadrightarrow \mu$.

In W, $\mu = \bigcup_{a \in A} f``X_a$.

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$, *then* $\mathrm{cf}(\mu) \leqslant |A|^W$.

Proof.

Let $\eta \in \mathrm{Ord}$ be such that $|X| \leqslant^* |A \times \eta|$, say $g\colon A \times \eta \twoheadrightarrow X$.

For $a \in A$, let $X_a = g``(\{a\} \times \eta)$. Note that X_a is well-orderable in M.

Let $\mu < \aleph^*(X)$, so there is a surjection $f\colon X \twoheadrightarrow \mu$.

In W, $\mu = \bigcup_{a \in A} f``X_a$. If $\mathrm{cf}(\mu) > |A|^W$, then there is $a \in A$ such that $|f``X_a| = |\mu|$.

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$, *then* $\mathrm{cf}(\mu) \leqslant |A|^W$*.*

Proof.

Let $\eta \in \mathrm{Ord}$ be such that $|X| \leqslant^* |A \times \eta|$, say $g\colon A \times \eta \twoheadrightarrow X$.

For $a \in A$, let $X_a = g``(\{a\} \times \eta)$. Note that X_a is well-orderable in M.

Let $\mu < \aleph^*(X)$, so there is a surjection $f\colon X \twoheadrightarrow \mu$.

In W, $\mu = \bigcup_{a \in A} f``X_a$. If $\mathrm{cf}(\mu) > |A|^W$, then there is $a \in A$ such that $|f``X_a| = |\mu|$.

Since $|f``X_a| = |\mu|$ we get $|\mu| \leqslant^* |X_a|$ (in M).

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$, *then* $\mathrm{cf}(\mu) \leqslant |A|^W$.

Proof.

Let $\eta \in \mathrm{Ord}$ be such that $|X| \leqslant^* |A \times \eta|$, say $g\colon A \times \eta \twoheadrightarrow X$.

For $a \in A$, let $X_a = g``(\{a\} \times \eta)$. Note that X_a is well-orderable in M.

Let $\mu < \aleph^*(X)$, so there is a surjection $f\colon X \twoheadrightarrow \mu$.

In W, $\mu = \bigcup_{a \in A} f``X_a$. If $\mathrm{cf}(\mu) > |A|^W$, then there is $a \in A$ such that $|f``X_a| = |\mu|$.

Since $|f``X_a| = |\mu|$ we get $|\mu| \leqslant^* |X_a|$ (in M).

X_a is well-orderable, so $|\mu| \leqslant^* |X_a| \implies |\mu| \leqslant |X_a|$.

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$, *then* $\mathrm{cf}(\mu) \leqslant |A|^W$.

Proof.

Let $\eta \in \mathrm{Ord}$ be such that $|X| \leqslant^* |A \times \eta|$, say $g\colon A \times \eta \twoheadrightarrow X$.

For $a \in A$, let $X_a = g``(\{a\} \times \eta)$. Note that X_a is well-orderable in M.

Let $\mu < \aleph^*(X)$, so there is a surjection $f\colon X \twoheadrightarrow \mu$.

In W, $\mu = \bigcup_{a \in A} f``X_a$. If $\mathrm{cf}(\mu) > |A|^W$, then there is $a \in A$ such that $|f``X_a| = |\mu|$.

Since $|f``X_a| = |\mu|$ we get $|\mu| \leqslant^* |X_a|$ (in M).

X_a is well-orderable, so $|\mu| \leqslant^* |X_a| \implies |\mu| \leqslant |X_a|$.

Hence $|\mu| \leqslant |X|$, and $\mu < \aleph(X)$. □

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$*, then* $\mathrm{cf}(\mu) \leqslant |A|^W$*.*

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$*, then* $\mathrm{cf}(\mu) \leqslant |A|^W$*.*

Corollary

If $|A|^W \leqslant \aleph(X) < \aleph^*(X)$

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$*, then* $\mathrm{cf}(\mu) \leqslant |A|^W$.

Corollary

If $|A|^W \leqslant \aleph(X) < \aleph^*(X)$ *then* $\mathrm{cf}(\aleph(X)) \leqslant |A|^W$

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$*, then* $\mathrm{cf}(\mu) \leqslant |A|^W$.

Corollary

If $|A|^W \leqslant \aleph(X) < \aleph^*(X)$ *then* $\mathrm{cf}(\aleph(X)) \leqslant |A|^W$ *and* $\aleph^*(X) = \aleph(X)^+$.

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$, *then* $\mathrm{cf}(\mu) \leqslant |A|^W$.

Corollary

If $|A|^W \leqslant \aleph(X) < \aleph^*(X)$ *then* $\mathrm{cf}(\aleph(X)) \leqslant |A|^W$ *and* $\aleph^*(X) = \aleph(X)^+$.

Proof.

Let $\lambda = \aleph(X)$.

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$, *then* $\mathrm{cf}(\mu) \leqslant |A|^W$.

Corollary

If $|A|^W \leqslant \aleph(X) < \aleph^*(X)$ *then* $\mathrm{cf}(\aleph(X)) \leqslant |A|^W$ *and* $\aleph^*(X) = \aleph(X)^+$.

Proof.

Let $\lambda = \aleph(X)$. Then $\aleph(X) \leqslant \lambda < \aleph^*(X)$,

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$, *then* $\mathrm{cf}(\mu) \leqslant |A|^W$.

Corollary

If $|A|^W \leqslant \aleph(X) < \aleph^*(X)$ *then* $\mathrm{cf}(\aleph(X)) \leqslant |A|^W$ *and* $\aleph^*(X) = \aleph(X)^+$.

Proof.

Let $\lambda = \aleph(X)$. Then $\aleph(X) \leqslant \lambda < \aleph^*(X)$, so $\mathrm{cf}(\lambda) \leqslant |A|^W$.

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$*, then* $\mathrm{cf}(\mu) \leqslant |A|^W$*.*

Corollary

If $|A|^W \leqslant \aleph(X) < \aleph^*(X)$ *then* $\mathrm{cf}(\aleph(X)) \leqslant |A|^W$ *and* $\aleph^*(X) = \aleph(X)^+$*.*

Proof.

Let $\lambda = \aleph(X)$. Then $\aleph(X) \leqslant \lambda < \aleph^*(X)$, so $\mathrm{cf}(\lambda) \leqslant |A|^W$.

If $\lambda^+ < \aleph^*(X)$

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$, *then* $\mathrm{cf}(\mu) \leqslant |A|^W$.

Corollary

If $|A|^W \leqslant \aleph(X) < \aleph^*(X)$ *then* $\mathrm{cf}(\aleph(X)) \leqslant |A|^W$ *and* $\aleph^*(X) = \aleph(X)^+$.

Proof.

Let $\lambda = \aleph(X)$. Then $\aleph(X) \leqslant \lambda < \aleph^*(X)$, so $\mathrm{cf}(\lambda) \leqslant |A|^W$.
If $\lambda^+ < \aleph^*(X)$ then $\aleph(X) \leqslant \lambda^+ < \aleph^*(X)$,

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$*, then* $\mathrm{cf}(\mu) \leqslant |A|^W$*.*

Corollary

If $|A|^W \leqslant \aleph(X) < \aleph^*(X)$ *then* $\mathrm{cf}(\aleph(X)) \leqslant |A|^W$ *and* $\aleph^*(X) = \aleph(X)^+$*.*

Proof.

Let $\lambda = \aleph(X)$. Then $\aleph(X) \leqslant \lambda < \aleph^*(X)$, so $\mathrm{cf}(\lambda) \leqslant |A|^W$.
If $\lambda^+ < \aleph^*(X)$ then $\aleph(X) \leqslant \lambda^+ < \aleph^*(X)$, so $\mathrm{cf}(\lambda^+) \leqslant |A|^W$.

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$*, then* $\operatorname{cf}(\mu) \leqslant |A|^W$*.*

Corollary

If $|A|^W \leqslant \aleph(X) < \aleph^*(X)$ *then* $\operatorname{cf}(\aleph(X)) \leqslant |A|^W$ *and* $\aleph^*(X) = \aleph(X)^+$*.*

Proof.

Let $\lambda = \aleph(X)$. Then $\aleph(X) \leqslant \lambda < \aleph^*(X)$, so $\operatorname{cf}(\lambda) \leqslant |A|^W$.
If $\lambda^+ < \aleph^*(X)$ then $\aleph(X) \leqslant \lambda^+ < \aleph^*(X)$, so $\operatorname{cf}(\lambda^+) \leqslant |A|^W$.
However, $\operatorname{cf}(\lambda^+) = \lambda^+ > |A|^W$,

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$, *then* $\operatorname{cf}(\mu) \leqslant |A|^W$.

Corollary

If $|A|^W \leqslant \aleph(X) < \aleph^*(X)$ *then* $\operatorname{cf}(\aleph(X)) \leqslant |A|^W$ *and* $\aleph^*(X) = \aleph(X)^+$.

Proof.

Let $\lambda = \aleph(X)$. Then $\aleph(X) \leqslant \lambda < \aleph^*(X)$, so $\operatorname{cf}(\lambda) \leqslant |A|^W$.
If $\lambda^+ < \aleph^*(X)$ then $\aleph(X) \leqslant \lambda^+ < \aleph^*(X)$, so $\operatorname{cf}(\lambda^+) \leqslant |A|^W$.
However, $\operatorname{cf}(\lambda^+) = \lambda^+ > |A|^W$, contradicting that $\lambda^+ < \aleph^*(X)$. □

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$, *then* $\mathrm{cf}(\mu) \leqslant |A|^W$.

Corollary

If $|A|^W \leqslant \aleph(X) < \aleph^*(X)$ *then* $\mathrm{cf}(\aleph(X)) \leqslant |A|^W$ *and* $\aleph^*(X) = \aleph(X)^+$.

Proof.

Let $\lambda = \aleph(X)$. Then $\aleph(X) \leqslant \lambda < \aleph^*(X)$, so $\mathrm{cf}(\lambda) \leqslant |A|^W$.
If $\lambda^+ < \aleph^*(X)$ then $\aleph(X) \leqslant \lambda^+ < \aleph^*(X)$, so $\mathrm{cf}(\lambda^+) \leqslant |A|^W$.
However, $\mathrm{cf}(\lambda^+) = \lambda^+ > |A|^W$, contradicting that $\lambda^+ < \aleph^*(X)$. □

This is why $\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc} \cup \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) = \omega\}$:

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$*, then* $\mathrm{cf}(\mu) \leqslant |A|^W$*.*

Corollary

If $|A|^W \leqslant \aleph(X) < \aleph^*(X)$ *then* $\mathrm{cf}(\aleph(X)) \leqslant |A|^W$ *and* $\aleph^*(X) = \aleph(X)^+$*.*

Proof.

Let $\lambda = \aleph(X)$. Then $\aleph(X) \leqslant \lambda < \aleph^*(X)$, so $\mathrm{cf}(\lambda) \leqslant |A|^W$.
If $\lambda^+ < \aleph^*(X)$ then $\aleph(X) \leqslant \lambda^+ < \aleph^*(X)$, so $\mathrm{cf}(\lambda^+) \leqslant |A|^W$.
However, $\mathrm{cf}(\lambda^+) = \lambda^+ > |A|^W$, contradicting that $\lambda^+ < \aleph^*(X)$. □

This is why $\mathrm{Spec}_\aleph(M) = \mathrm{Suc} \cup \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) = \omega\}$: If $\aleph(X) < \aleph^*(X)$

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$*, then* $\mathrm{cf}(\mu) \leqslant |A|^W$*.*

Corollary

If $|A|^W \leqslant \aleph(X) < \aleph^*(X)$ *then* $\mathrm{cf}(\aleph(X)) \leqslant |A|^W$ *and* $\aleph^*(X) = \aleph(X)^+$*.*

Proof.

Let $\lambda = \aleph(X)$. Then $\aleph(X) \leqslant \lambda < \aleph^*(X)$, so $\mathrm{cf}(\lambda) \leqslant |A|^W$.
If $\lambda^+ < \aleph^*(X)$ then $\aleph(X) \leqslant \lambda^+ < \aleph^*(X)$, so $\mathrm{cf}(\lambda^+) \leqslant |A|^W$.
However, $\mathrm{cf}(\lambda^+) = \lambda^+ > |A|^W$, contradicting that $\lambda^+ < \aleph^*(X)$. □

This is why $\mathrm{Spec}_\aleph(M) = \mathrm{Suc} \cup \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) = \omega\}$: If $\aleph(X) < \aleph^*(X)$ then $\mathrm{cf}(\aleph(X)) = \omega$ and $\aleph^*(X) = \aleph(X)^+$.

Let $M \vDash \mathsf{SVC}(A)$. Then:

Let $M \vDash \mathsf{SVC}(A)$. Then:

1. $\mathrm{Suc} = \{\langle \lambda^+, \lambda^+ \rangle \mid \lambda \text{ a cardinal}\} \subseteq \mathrm{Spec}_{\aleph}(M)$.

Let $M \vDash \mathsf{SVC}(A)$. Then:

1. $\mathrm{Suc} = \{\langle \lambda^+, \lambda^+ \rangle \mid \lambda \text{ a cardinal}\} \subseteq \mathrm{Spec}_{\aleph}(M)$.
2. If $\aleph(X)^+ < \aleph^*(X)$ then $\aleph(X) < |A|^W$.

Let $M \vDash \mathsf{SVC}(A)$. Then:

1. $\mathrm{Suc} = \{\langle \lambda^+, \lambda^+ \rangle \mid \lambda \text{ a cardinal}\} \subseteq \mathrm{Spec}_{\aleph}(M)$.
2. If $\aleph(X)^+ < \aleph^*(X)$ then $\aleph(X) < |A|^W$.
3. $C = \{\mu \mid (\exists X)\, \mathrm{cf}(\aleph(X)) = \mu \wedge \aleph(X) < \aleph^*(X)\} \subseteq \{\mu \mid \mu \leqslant |A|^W\}$ is a **set**.

Let $M \vDash \mathsf{SVC}(A)$. Then:

1. $\mathrm{Suc} = \{\langle\lambda^+, \lambda^+\rangle \mid \lambda \text{ a cardinal}\} \subseteq \mathrm{Spec}_\aleph(M)$.
2. If $\aleph(X)^+ < \aleph^*(X)$ then $\aleph(X) < |A|^W$.
3. $C = \{\mu \mid (\exists X)\,\mathrm{cf}(\aleph(X)) = \mu \wedge \aleph(X) < \aleph^*(X)\} \subseteq \{\mu \mid \mu \leqslant |A|^W\}$ is a **set**.
4. If $\mu \in C$ then for **all large enough** λ such that $\mathrm{cf}(\lambda) = \mu$ there is X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \lambda^+$.

Let $M \vDash \text{SVC}(A)$. Then:

1. $\mathrm{Suc} = \{\langle\lambda^+, \lambda^+\rangle \mid \lambda$ a cardinal$\} \subseteq \mathrm{Spec}_{\aleph}(M)$.
2. If $\aleph(X)^+ < \aleph^*(X)$ then $\aleph(X) < |A|^W$.
3. $C = \{\mu \mid (\exists X)\,\mathrm{cf}(\aleph(X)) = \mu \wedge \aleph(X) < \aleph^*(X)\} \subseteq \{\mu \mid \mu \leqslant |A|^W\}$ is a **set**.
4. If $\mu \in C$ then for **all large enough** λ such that $\mathrm{cf}(\lambda) = \mu$ there is X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \lambda^+$.
   So there is χ such that, for all $\lambda \geqslant \chi$,

Let $M \vDash$ SVC(A). Then:

1. $\mathrm{Suc} = \{\langle \lambda^+, \lambda^+ \rangle \mid \lambda$ a cardinal$\} \subseteq \mathrm{Spec}_\aleph(M)$.
2. If $\aleph(X)^+ < \aleph^*(X)$ then $\aleph(X) < |A|^W$.
3. $C = \{\mu \mid (\exists X)\, \mathrm{cf}(\aleph(X)) = \mu \wedge \aleph(X) < \aleph^*(X)\} \subseteq \{\mu \mid \mu \leqslant |A|^W\}$ is a **set**.
4. If $\mu \in C$ then for **all large enough** λ such that $\mathrm{cf}(\lambda) = \mu$ there is X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \lambda^+$.
   So there is χ such that, for all $\lambda \geqslant \chi$,
   $(\exists X)\lambda = \aleph(X) < \aleph^*(X)$ **if and only if** $\mathrm{cf}(\lambda) \in C$ and $\aleph^*(X) = \lambda^+$.

Let $M \vDash \mathsf{SVC}(A)$. Then:

1. $\mathrm{Suc} = \{\langle\lambda^+, \lambda^+\rangle \mid \lambda$ a cardinal$\} \subseteq \mathrm{Spec}_{\aleph}(M)$.
2. If $\aleph(X)^+ < \aleph^*(X)$ then $\aleph(X) < |A|^W$.
3. $C = \{\mu \mid (\exists X)\,\mathrm{cf}(\aleph(X)) = \mu \wedge \aleph(X) < \aleph^*(X)\} \subseteq \{\mu \mid \mu \leqslant |A|^W\}$ is a **set**.
4. If $\mu \in C$ then for **all large enough** λ such that $\mathrm{cf}(\lambda) = \mu$ there is X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \lambda^+$.
   So there is χ such that, for all $\lambda \geqslant \chi$,

   $(\exists X)\lambda = \aleph(X) < \aleph^*(X)$ **if and only if** $\mathrm{cf}(\lambda) \in C$ and $\aleph^*(X) = \lambda^+$.

$$\mathrm{Spec}_{\aleph}(M) = \bigcup \left\{ \vphantom{\int} \right.$$

Let $M \vDash \mathsf{SVC}(A)$. Then:

1. $\mathrm{Suc} = \{\langle\lambda^+, \lambda^+\rangle \mid \lambda \text{ a cardinal}\} \subseteq \mathrm{Spec}_{\aleph}(M)$.
2. If $\aleph(X)^+ < \aleph^*(X)$ then $\aleph(X) < |A|^W$.
3. $C = \{\mu \mid (\exists X)\, \mathrm{cf}(\aleph(X)) = \mu \wedge \aleph(X) < \aleph^*(X)\} \subseteq \{\mu \mid \mu \leqslant |A|^W\}$ is a **set**.
4. If $\mu \in C$ then for **all large enough** λ such that $\mathrm{cf}(\lambda) = \mu$ there is X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \lambda^+$.
   So there is χ such that, for all $\lambda \geqslant \chi$,
   $(\exists X)\lambda = \aleph(X) < \aleph^*(X)$ **if and only if** $\mathrm{cf}(\lambda) \in C$ and $\aleph^*(X) = \lambda^+$.

$$\mathrm{Spec}_{\aleph}(M) = \bigcup \begin{cases} \mathrm{Suc} = \{\langle\lambda^+, \lambda^+\rangle \mid \lambda \in \mathrm{Card}\} \\ \\ \\ \end{cases}$$

Let $M \vDash \mathsf{SVC}(A)$. Then:

1. $\mathrm{Suc} = \{\langle\lambda^+, \lambda^+\rangle \mid \lambda \text{ a cardinal}\} \subseteq \mathrm{Spec}_{\aleph}(M)$.
2. If $\aleph(X)^+ < \aleph^*(X)$ then $\aleph(X) < |A|^W$.
3. $C = \{\mu \mid (\exists X)\, \mathrm{cf}(\aleph(X)) = \mu \wedge \aleph(X) < \aleph^*(X)\} \subseteq \{\mu \mid \mu \leqslant |A|^W\}$ is a **set**.
4. If $\mu \in C$ then for **all large enough** λ such that $\mathrm{cf}(\lambda) = \mu$ there is X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \lambda^+$.
   So there is χ such that, for all $\lambda \geqslant \chi$,

   $$(\exists X)\lambda = \aleph(X) < \aleph^*(X) \textbf{ if and only if } \mathrm{cf}(\lambda) \in C \text{ and } \aleph^*(X) = \lambda^+.$$

$$\mathrm{Spec}_{\aleph}(M) = \bigcup \begin{cases} \mathrm{Suc} = \{\langle\lambda^+, \lambda^+\rangle \mid \lambda \in \mathrm{Card}\} \\ \mathfrak{D} \subseteq \{\langle\lambda, \kappa\rangle \quad \mid \lambda \leqslant \kappa,\ \lambda < |A|^W\} \\ \\ \\ \end{cases}$$

Let $M \vDash \mathsf{SVC}(A)$. Then:

1. $\mathrm{Suc} = \{\langle\lambda^+, \lambda^+\rangle \mid \lambda$ a cardinal$\} \subseteq \mathrm{Spec}_{\aleph}(M)$.
2. If $\aleph(X)^+ < \aleph^*(X)$ then $\aleph(X) < |A|^W$.
3. $C = \{\mu \mid (\exists X)\,\mathrm{cf}(\aleph(X)) = \mu \wedge \aleph(X) < \aleph^*(X)\} \subseteq \{\mu \mid \mu \leqslant |A|^W\}$ is a **set**.
4. If $\mu \in C$ then for **all large enough** λ such that $\mathrm{cf}(\lambda) = \mu$ there is X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \lambda^+$.
   So there is χ such that, for all $\lambda \geqslant \chi$,

   $$(\exists X)\lambda = \aleph(X) < \aleph^*(X) \text{ **if and only if** } \mathrm{cf}(\lambda) \in C \text{ and } \aleph^*(X) = \lambda^+.$$

$$\mathrm{Spec}_{\aleph}(M) = \bigcup \begin{cases} \mathrm{Suc} = \{\langle\lambda^+, \lambda^+\rangle \mid \lambda \in \mathrm{Card}\} \\ \mathfrak{D} \subseteq \{\langle\lambda, \kappa\rangle \mid \lambda \leqslant \kappa,\ \lambda < |A|^W\} \\ \mathfrak{C} \subseteq \{\langle\lambda, \lambda^+\rangle \mid \mathrm{cf}(\lambda) \in C, \lambda < \chi\} \end{cases}$$

Let $M \vDash \mathsf{SVC}(A)$. Then:

1. $\mathrm{Suc} = \{\langle\lambda^+, \lambda^+\rangle \mid \lambda$ a cardinal$\} \subseteq \mathrm{Spec}_{\aleph}(M)$.
2. If $\aleph(X)^+ < \aleph^*(X)$ then $\aleph(X) < |A|^W$.
3. $C = \{\mu \mid (\exists X)\,\mathrm{cf}(\aleph(X)) = \mu \wedge \aleph(X) < \aleph^*(X)\} \subseteq \{\mu \mid \mu \leqslant |A|^W\}$ is a **set**.
4. If $\mu \in C$ then for **all large enough** λ such that $\mathrm{cf}(\lambda) = \mu$ there is X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \lambda^+$.
   So there is χ such that, for all $\lambda \geqslant \chi$,
   $$(\exists X)\lambda = \aleph(X) < \aleph^*(X) \text{ **if and only if** } \mathrm{cf}(\lambda) \in C \text{ and } \aleph^*(X) = \lambda^+.$$

$$\mathrm{Spec}_{\aleph}(M) = \bigcup \begin{cases} \mathrm{Suc} = \{\langle\lambda^+, \lambda^+\rangle \mid \lambda \in \mathrm{Card}\} \\ \mathfrak{D} \subseteq \{\langle\lambda, \kappa\rangle \quad \mid \lambda \leqslant \kappa,\ \lambda < |A|^W\} \\ \mathfrak{C} \subseteq \{\langle\lambda, \lambda^+\rangle \quad \mid \mathrm{cf}(\lambda) \in C, \lambda < \chi\} \\ \mathfrak{U} = \{\langle\lambda, \lambda^+\rangle \quad \mid \mathrm{cf}(\lambda) \in C, \lambda \geqslant \chi\}. \end{cases}$$

Open questions

1. How much of a gap can we have in the spectrum?

Open questions

1. How much of a gap can we have in the spectrum? E.g., is ZF consistent with

$$\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc} \cup \{\langle \omega_1, \omega_2 \rangle\} \cup \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) = \omega_1,\ \lambda \geqslant \kappa\}$$

   for arbitrarily large κ?

Open questions

1. How much of a gap can we have in the spectrum? E.g., is ZF consistent with

$$\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc} \cup \{\langle \omega_1, \omega_2 \rangle\} \cup \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) = \omega_1,\ \lambda \geqslant \kappa\}$$

   for arbitrarily large κ?

2. Lindenbaum numbers are not quite multiplicative:

$$\aleph^*(A) \times \aleph^*(B) \leqslant \aleph^*(A \times B) \leqslant (\aleph^*(A) \times \aleph^*(B))^+.$$

Open questions

1. How much of a gap can we have in the spectrum? E.g., is ZF consistent with

$$\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc} \cup \{\langle \omega_1, \omega_2 \rangle\} \cup \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) = \omega_1,\ \lambda \geqslant \kappa\}$$

   for arbitrarily large κ?

2. Lindenbaum numbers are not quite multiplicative:

$$\aleph^*(A) \times \aleph^*(B) \leqslant \aleph^*(A \times B) \leqslant (\aleph^*(A) \times \aleph^*(B))^+.$$

   When is either bound reached? When is either bound *possible* to reach?

Open questions

1. How much of a gap can we have in the spectrum? E.g., is ZF consistent with

$$\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc} \cup \{\langle \omega_1, \omega_2 \rangle\} \cup \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) = \omega_1,\ \lambda \geqslant \kappa\}$$

   for arbitrarily large κ?
2. Lindenbaum numbers are not quite multiplicative:

$$\aleph^*(A) \times \aleph^*(B) \leqslant \aleph^*(A \times B) \leqslant (\aleph^*(A) \times \aleph^*(B))^+.$$

   When is either bound reached? When is either bound *possible* to reach?
3. If $M \vDash$ SVC, when can AC be forced without collapsing cardinals?

Bibliography I

[Bla79] Andreas Blass. “Injectivity, projectivity, and the axiom of choice”. In: *Trans. Amer. Math. Soc.* 255 (1979), pp. 31–59. ISSN: 0002-9947,1088-6850.

[Har15] F. Hartogs. “Über das Problem der Wohlordnung”. German. In: *Math. Ann.* 76.4 (1915), pp. 438–443. ISSN: 0025-5831,1432-1807.

[KRS23] Asaf Karagila and Calliope Ryan-Smith. “Which pairs of cardinals can be Hartogs and Lindenbaum numbers of a set?” In: *arXiv* 2309.11409 (2023).

[LT26] A. Lindenbaum and A. Tarski. “Communication sur les recherches de la théorie des ensembles.”. French. In: *C. R. Soc. Sci. Varsovie, Cl. III* 19 (1926), pp. 299–330.

Bibliography II

[RS24] Calliope Ryan-Smith. "The Hartogs–Lindenbaum spectrum of symmetric extensions". In: *Math. Log. Quart.* 70.2 (2024), pp. 210–223.

[Usu21] Toshimichi Usuba. "Choiceless Löwenheim-Skolem property and uniform definability of grounds". In: *Advances in mathematical logic*. Vol. 369. Springer Proc. Math. Stat. Springer, Singapore, 2021, pp. 161–179. ISBN: 978-981-16-4172-5; 978-981-16-4173-2.

academic.calliope.mx c.Ryan-Smith@leeds.ac.uk