The Hartogs–Lindenbaum Spectrum

Calliope Ryan-Smith

Univeristy of Leeds

British Logic Colloquium 2024

5th September 2024

c.Ryan-Smith@leeds.ac.uk

https://academic.calliope.mx

How big is a set?

How big is a set?

Theorem (Zermelo, *Well-ordering principle*)

(ZFC) *For all X there is an ordinal α such that $|X| = |\alpha|$.*[a]

How big is a set?

Theorem (Zermelo, *Well-ordering principle*)

(ZFC) For all X there is an ordinal α such that $|X| = |\alpha|$.[a]

[a] There is a bijection $\alpha \to X$. $|A| \leqslant |B|$ if there is an injection $A \to B$. $|A| \leqslant^* |B|$ if there is a surjection $B \to A$ (or $A = \emptyset$).

How big is a set?

Theorem (Zermelo, *Well-ordering principle*)

(ZFC) For all X there is an ordinal α such that $|X| = |\alpha|$.[a]

[a] There is a bijection $\alpha \to X$. $|A| \leqslant |B|$ if there is an injection $A \to B$. $|A| \leqslant^* |B|$ if there is a surjection $B \to A$ (or $A = \emptyset$).

So $|X| = \min\{\alpha \in \mathrm{Ord} \mid |X| = |\alpha|\}$. In ZF, this is no longer an option.

How big is a set?

Theorem (Zermelo, *Well-ordering principle*)

(ZFC) For all X there is an ordinal α such that $|X| = |\alpha|$.[a]

---

[a] There is a bijection $\alpha \to X$. $|A| \leqslant |B|$ if there is an injection $A \to B$. $|A| \leqslant^* |B|$ if there is a surjection $B \to A$ (or $A = \emptyset$).

So $|X| = \min\{\alpha \in \mathrm{Ord} \mid |X| = |\alpha|\}$. In ZF, this is no longer an option.

Example

In *Cohen's first model* of ZF $+ \neg$AC there is an infinite set A such that there is no injection $\omega \to A$, but there is a surjection $A \to \omega$.

Approximate size

Approximate size

Definition (Hartogs/Lindenbaum number)

The *Hartogs number* of X is $\aleph(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq |X|\}$.

Approximate size

Definition (Hartogs/Lindenbaum number)

The *Hartogs number* of X is $\aleph(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq |X|\}$.

The *Lindenbaum number* of X is $\aleph^*(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq^* |X|\}$.

Approximate size

Definition (Hartogs/Lindenbaum number)

The *Hartogs number* of X is $\aleph(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant |X|\}$.

The *Lindenbaum number* of X is $\aleph^*(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant^* |X|\}$.

Fact/exercise

(ZF) For all X, $\aleph(X)$ and $\aleph^*(X)$ exist, and $\aleph(X) \leqslant \aleph^*(X)$.

Approximate size

Definition (Hartogs/Lindenbaum number)

The *Hartogs number* of X is $\aleph(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant |X|\}$.
The *Lindenbaum number* of X is $\aleph^*(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant^* |X|\}$.

Fact/exercise

(ZF) For all X, $\aleph(X)$ and $\aleph^*(X)$ exist, and $\aleph(X) \leqslant \aleph^*(X)$.

So in ZFC, $\aleph(X) = \aleph^*(X) = |X|^+$ for all X.

Approximate size

Definition (Hartogs/Lindenbaum number)

The *Hartogs number* of X is $\aleph(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant |X|\}$.
The *Lindenbaum number* of X is $\aleph^*(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant^* |X|\}$.

Fact/exercise

(ZF) For all X, $\aleph(X)$ and $\aleph^*(X)$ exist, and $\aleph(X) \leqslant \aleph^*(X)$.

So in ZFC, $\aleph(X) = \aleph^*(X) = |X|^+$ for all X. In Cohen's first model, $\aleph(A) = \omega$ and $\aleph^*(A) = \omega_1$.

Approximate size

Definition (Hartogs/Lindenbaum number)

The *Hartogs number* of X is $\aleph(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant |X|\}$.
The *Lindenbaum number* of X is $\aleph^*(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant^* |X|\}$.

Fact/exercise

(ZF) For all X, $\aleph(X)$ and $\aleph^*(X)$ exist, and $\aleph(X) \leqslant \aleph^*(X)$.

So in ZFC, $\aleph(X) = \aleph^*(X) = |X|^+$ for all X. In Cohen's first model, $\aleph(A) = \omega$ and $\aleph^*(A) = \omega_1$.

Example (Cohen's first model)

There is X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$ if and only if:

Approximate size

Definition (Hartogs/Lindenbaum number)

The *Hartogs number* of X is $\aleph(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant |X|\}$.
The *Lindenbaum number* of X is $\aleph^*(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant^* |X|\}$.

Fact/exercise

(ZF) For all X, $\aleph(X)$ and $\aleph^*(X)$ exist, and $\aleph(X) \leqslant \aleph^*(X)$.

So in ZFC, $\aleph(X) = \aleph^*(X) = |X|^+$ for all X. In Cohen's first model, $\aleph(A) = \omega$ and $\aleph^*(A) = \omega_1$.

Example (Cohen's first model)

There is X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$ if and only if:

1. $\lambda = \kappa = \eta^+$ for some cardinal η; or

Approximate size

Definition (Hartogs/Lindenbaum number)

The *Hartogs number* of X is $\aleph(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant |X|\}$.
The *Lindenbaum number* of X is $\aleph^*(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant^* |X|\}$.

Fact/exercise

(ZF) For all X, $\aleph(X)$ and $\aleph^*(X)$ exist, and $\aleph(X) \leqslant \aleph^*(X)$.

So in ZFC, $\aleph(X) = \aleph^*(X) = |X|^+$ for all X. In Cohen's first model, $\aleph(A) = \omega$ and $\aleph^*(A) = \omega_1$.

Example (Cohen's first model)

There is X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$ if and only if:

1. $\lambda = \kappa = \eta^+$ for some cardinal η; or
2. $\kappa = \lambda^+$ and $\mathrm{cf}(\lambda) = \omega$.

Hartogs–Lindenbaum spectrum

Hartogs–Lindenbaum spectrum

Definition

For a model $M \vDash \mathrm{ZF}$, the *Hartogs–Lindenbaum spectrum* of M is

$$\mathrm{Spec}_{\aleph}(M) = \{\langle \lambda, \kappa \rangle \mid (\exists X \in M)\aleph(X) = \lambda \text{ and } \aleph^*(X) = \kappa\}.$$

Hartogs–Lindenbaum spectrum

Definition

For a model $M \vDash$ ZF, the *Hartogs–Lindenbaum spectrum* of M is

$$\mathrm{Spec}_{\aleph}(M) = \{\langle \lambda, \kappa \rangle \mid (\exists X \in M)\aleph(X) = \lambda \text{ and } \aleph^*(X) = \kappa\}.$$

Since $\aleph(\lambda) = \aleph^*(\lambda) = \lambda^+$ for all λ, $\mathrm{Suc} = \{\langle \lambda^+, \lambda^+ \rangle \mid \lambda \in \mathrm{Card}\} \subseteq \mathrm{Spec}_{\aleph}(M)$.

Hartogs–Lindenbaum spectrum

Definition

For a model $M \vDash \mathrm{ZF}$, the *Hartogs–Lindenbaum spectrum* of M is

$$\mathrm{Spec}_{\aleph}(M) = \{\langle \lambda, \kappa \rangle \mid (\exists X \in M)\aleph(X) = \lambda \text{ and } \aleph^*(X) = \kappa\}.$$

Since $\aleph(\lambda) = \aleph^*(\lambda) = \lambda^+$ for all λ, $\mathrm{Suc} = \{\langle \lambda^+, \lambda^+ \rangle \mid \lambda \in \mathrm{Card}\} \subseteq \mathrm{Spec}_{\aleph}(M)$.

Example

Letting M be Cohen's first model,

$$\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc} \cup \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) = \omega\}.$$

Symmetric extensions

Cohen's first model is a *symmetric extension*.

Symmetric extensions

Cohen's first model is a *symmetric extension*.

Symmetric extensions

Cohen's first model is a *symmetric extension*.

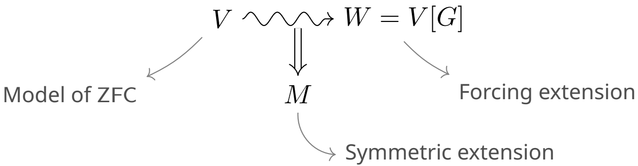
$$V \rightsquigarrow W = V[G]$$

Model of ZFC

Forcing extension

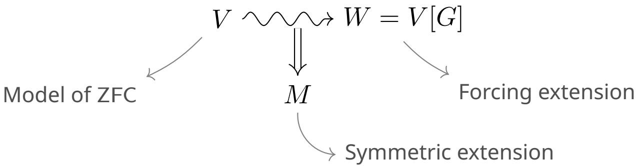
Symmetric extensions

Cohen's first model is a *symmetric extension*.



Symmetric extensions

Cohen's first model is a *symmetric extension*.

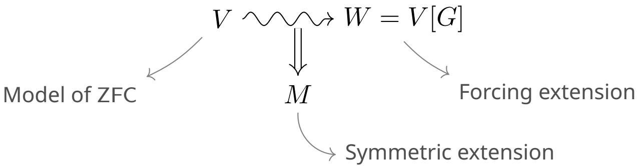


Theorem (Blass/Usuba, [Bla79; Usu21])

M *is a symmetric extension of a model of* ZFC *if and only if it is a model of* small violations of choice*: There is an* ***injective seed*** A *such that, for all* X*, there is* $\eta \in \mathrm{Ord}$ *and an injection* $X \to A \times \eta$*.*

Symmetric extensions

Cohen's first model is a *symmetric extension*.



Theorem (Blass/Usuba, [Bla79; Usu21])

M is a symmetric extension of a model of ZFC *if and only if it is a model of* small violations of choice*: There is an* ***injective seed*** *A such that, for all X, there is $\eta \in \mathrm{Ord}$ and an injection $X \to A \times \eta$.*

For technical simplicity, let us assume that M and W agree on the cardinality and cofinalities of all ordinals.

$M \subseteq W$, $W \vDash$ ZFC, $M \vDash (\forall X)(\exists \eta \in \mathrm{Ord})(|X| \leqslant |A \times \eta|)$.

$M \subseteq W$, $W \vDash$ ZFC, $M \vDash (\forall X)(\exists \eta \in \mathrm{Ord})(|X| \leqslant |A \times \eta|)$.

Theorem (R.S. [RS24])

(In M) For all X, if $\aleph(X) \leqslant \mu < \aleph^*(X)$, then $\mathrm{cf}(\mu) \leqslant |A|^W$.

$M \subseteq W$, $W \vDash$ ZFC, $M \vDash (\forall X)(\exists \eta \in \mathrm{Ord})(|X| \leqslant |A \times \eta|)$.

Theorem (R.S. [RS24])

(In M) For all X, if $\aleph(X) \leqslant \mu < \aleph^*(X)$, then $\mathrm{cf}(\mu) \leqslant |A|^W$.

Proof.

Let $f : X \hookrightarrow A \times \eta$.

$M \subseteq W$, $W \vDash$ ZFC, $M \vDash (\forall X)(\exists \eta \in \mathrm{Ord})(|X| \leqslant |A \times \eta|)$.

Theorem (R.S. [RS24])

(In M) For all X, if $\aleph(X) \leqslant \mu < \aleph^*(X)$, then $\mathrm{cf}(\mu) \leqslant |A|^W$.

Proof.

Let $f : X \hookrightarrow A \times \eta$. For all $b \in A$, let $X_b = f^{-1}(\{b\} \times \eta)$.

$M \subseteq W$, $W \vDash$ ZFC, $M \vDash (\forall X)(\exists \eta \in \mathrm{Ord})(|X| \leqslant |A \times \eta|)$.

Theorem (R.S. [RS24])

(In M) For all X, if $\aleph(X) \leqslant \mu < \aleph^*(X)$, then $\mathrm{cf}(\mu) \leqslant |A|^W$.

Proof.

Let $f\colon X \hookrightarrow A \times \eta$. For all $b \in A$, let $X_b = f^{-1}(\{b\} \times \eta)$.

Let $\mu < \aleph^*(X)$, so there is $g\colon X \twoheadrightarrow \mu$. Then $\mu = \bigcup_{b \in A} g``X_b$.

$M \subseteq W$, $W \vDash$ ZFC, $M \vDash (\forall X)(\exists \eta \in \mathrm{Ord})(|X| \leqslant |A \times \eta|)$.

Theorem (R.S. [RS24])

(In M) For all X, if $\aleph(X) \leqslant \mu < \aleph^*(X)$, then $\mathrm{cf}(\mu) \leqslant |A|^W$.

Proof.

Let $f\colon X \hookrightarrow A \times \eta$. For all $b \in A$, let $X_b = f^{-1}(\{b\} \times \eta)$.

Let $\mu < \aleph^*(X)$, so there is $g\colon X \twoheadrightarrow \mu$. Then $\mu = \bigcup_{b \in A} g``X_b$.

In W, if $\mathrm{cf}(\mu) > |A|$, then there is $b \in A$ such that $|g``X_b| = \mu$.

$M \subseteq W$, $W \vDash$ ZFC, $M \vDash (\forall X)(\exists \eta \in \mathrm{Ord})(|X| \leqslant |A \times \eta|)$.

Theorem (R.S. [RS24])

(In M) For all X, if $\aleph(X) \leqslant \mu < \aleph^*(X)$, then $\mathrm{cf}(\mu) \leqslant |A|^W$.

Proof.

Let $f\colon X \hookrightarrow A \times \eta$. For all $b \in A$, let $X_b = f^{-1}(\{b\} \times \eta)$.

Let $\mu < \aleph^*(X)$, so there is $g\colon X \twoheadrightarrow \mu$. Then $\mu = \bigcup_{b \in A} g``X_b$.

In W, if $\mathrm{cf}(\mu) > |A|$, then there is $b \in A$ such that $|g``X_b| = \mu$.

X_b is well-orderable in M, so $|\mu| \leqslant^* |X_b| \implies |\mu| \leqslant |X_b|$.

Thus $\mu < \aleph(X)$. □

$M \subseteq W$, $W \vDash$ ZFC, $M \vDash (\forall X)(\exists \eta \in \mathrm{Ord})(|X| \leqslant |A \times \eta|)$.

Theorem (R.S. [RS24])

(In M) For all X, if $\aleph(X) \leqslant \mu < \aleph^*(X)$, then $\mathrm{cf}(\mu) \leqslant |A|^W$.

Proof.

Let $f\colon X \hookrightarrow A \times \eta$. For all $b \in A$, let $X_b = f^{-1}(\{b\} \times \eta)$.

Let $\mu < \aleph^*(X)$, so there is $g\colon X \twoheadrightarrow \mu$. Then $\mu = \bigcup_{b \in A} g``X_b$.

In W, if $\mathrm{cf}(\mu) > |A|$, then there is $b \in A$ such that $|g``X_b| = \mu$.

X_b is well-orderable in M, so $|\mu| \leqslant^* |X_b| \implies |\mu| \leqslant |X_b|$.

Thus $\mu < \aleph(X)$. □

Corollary

If $\aleph(X) \geqslant |A|^W$ then $\aleph^*(X) = \aleph(X)$ or $\aleph(X)^+$.

$M \subseteq W$, $W \vDash$ ZFC, $M \vDash (\forall X)(\exists \eta \in \mathrm{Ord})(|X| \leqslant |A \times \eta|)$.

Theorem (R.S. [RS24])

(In M) For all X, if $\aleph(X) \leqslant \mu < \aleph^*(X)$, then $\mathrm{cf}(\mu) \leqslant |A|^W$.

Proof.

Let $f\colon X \hookrightarrow A \times \eta$. For all $b \in A$, let $X_b = f^{-1}(\{b\} \times \eta)$.

Let $\mu < \aleph^*(X)$, so there is $g\colon X \twoheadrightarrow \mu$. Then $\mu = \bigcup_{b \in A} g``X_b$.

In W, if $\mathrm{cf}(\mu) > |A|$, then there is $b \in A$ such that $|g``X_b| = \mu$.

X_b is well-orderable in M, so $|\mu| \leqslant^* |X_b| \implies |\mu| \leqslant |X_b|$.

Thus $\mu < \aleph(X)$. □

Corollary

If $\aleph(X) \geqslant |A|^W$ then $\aleph^*(X) = \aleph(X)$ or $\aleph(X)^+$.

Proof.

$\mathrm{cf}(\aleph(X)^+) = \aleph(X)^+ > |A|^W$,

$M \subseteq W$, $W \vDash$ ZFC, $M \vDash (\forall X)(\exists \eta \in \mathrm{Ord})(|X| \leqslant |A \times \eta|)$.

Theorem (R.S. [RS24])

(In M) For all X, if $\aleph(X) \leqslant \mu < \aleph^*(X)$, then $\mathrm{cf}(\mu) \leqslant |A|^W$.

Proof.

Let $f\colon X \hookrightarrow A \times \eta$. For all $b \in A$, let $X_b = f^{-1}(\{b\} \times \eta)$.

Let $\mu < \aleph^*(X)$, so there is $g\colon X \twoheadrightarrow \mu$. Then $\mu = \bigcup_{b \in A} g``X_b$.

In W, if $\mathrm{cf}(\mu) > |A|$, then there is $b \in A$ such that $|g``X_b| = \mu$.

X_b is well-orderable in M, so $|\mu| \leqslant^* |X_b| \implies |\mu| \leqslant |X_b|$.

Thus $\mu < \aleph(X)$. □

Corollary

If $\aleph(X) \geqslant |A|^W$ then $\aleph^*(X) = \aleph(X)$ or $\aleph(X)^+$.

Proof.

$\mathrm{cf}(\aleph(X)^+) = \aleph(X)^+ > |A|^W$, so $\neg\,(\aleph(X) \leqslant \aleph(X)^+ < \aleph^*(X))$. □

Theorem (R.S., (SVC))

$$\mathrm{Spec}_{\aleph}(M) \subseteq \bigcup \left\{ \right.$$

Theorem (R.S., (SVC))

$$\mathrm{Spec}_{\aleph}(M) \subseteq \bigcup \begin{cases} \mathrm{Suc} = \{\langle \lambda^+, \lambda^+ \rangle \mid \lambda \textit{ a cardinal}\} \\ \\ \\ \end{cases}$$

Theorem (R.S., (SVC))

$$\mathrm{Spec}_{\aleph}(M) \subseteq \bigcup \begin{cases} \mathrm{Suc} = \{\langle \lambda^+, \lambda^+ \rangle \mid \lambda \textit{ a cardinal}\} \\ \{\langle \lambda, \kappa \rangle \mid \lambda \leqslant \kappa \leqslant |A|^W\} \\ \end{cases}$$

Theorem (R.S., (SVC))

$$\mathrm{Spec}_{\aleph}(M) \subseteq \bigcup \begin{cases} \mathrm{Suc} = \{\langle \lambda^+, \lambda^+ \rangle \mid \lambda \text{ a cardinal}\} \\ \{\langle \lambda, \kappa \rangle \mid \lambda \leqslant \kappa \leqslant |A|^W\} \\ \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) \leqslant |A|^W\} \quad \leftarrow \text{"oblate cardinals"} \end{cases}$$

Theorem (R.S., (SVC))

$$\mathrm{Spec}_{\aleph}(M) \subseteq \bigcup \begin{cases} \mathrm{Suc} = \{\langle \lambda^+, \lambda^+ \rangle \mid \lambda \textit{ a cardinal}\} \\ \{\langle \lambda, \kappa \rangle \mid \lambda \leqslant \kappa \leqslant |A|^W\} \\ \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) \leqslant |A|^W\} \quad \leftarrow \textit{“oblate cardinals”} \end{cases}$$

The oblate cardinals eventually stabilise: If $\aleph(X) = \lambda < \aleph^*(X)$ then we can ‘lift’ X to produce Y such that $\aleph(Y) = \kappa$, $\aleph^*(Y) = \kappa^+$, and $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$.

Theorem (R.S., (SVC))

$$\mathrm{Spec}_{\aleph}(M) \subseteq \bigcup \begin{cases} \mathrm{Suc} = \{\langle \lambda^+, \lambda^+ \rangle \mid \lambda \textit{ a cardinal}\} \\ \{\langle \lambda, \kappa \rangle \mid \lambda \leqslant \kappa \leqslant |A|^W\} \\ \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) \leqslant |A|^W\} \quad \leftarrow \textit{“oblate cardinals”} \end{cases}$$

The oblate cardinals eventually stabilise: If $\aleph(X) = \lambda < \aleph^*(X)$ then we can ‘lift’ X to produce Y such that $\aleph(Y) = \kappa$, $\aleph^*(Y) = \kappa^+$, and $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$.

In fact, for *all* large enough κ such that $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$ such a Y can be constructed.

Theorem (R.S., (SVC))

$$\mathrm{Spec}_{\aleph}(M) \subseteq \bigcup \begin{cases} \mathrm{Suc} = \{\langle \lambda^+, \lambda^+ \rangle \mid \lambda \text{ a cardinal}\} \\ \{\langle \lambda, \kappa \rangle \mid \lambda \leqslant \kappa \leqslant |A|^W\} \\ \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) \leqslant |A|^W\} \quad \leftarrow \text{"oblate cardinals"} \end{cases}$$

The oblate cardinals eventually stabilise: If $\aleph(X) = \lambda < \aleph^*(X)$ then we can 'lift' X to produce Y such that $\aleph(Y) = \kappa$, $\aleph^*(Y) = \kappa^+$, and $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$.

In fact, for *all* large enough κ such that $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$ such a Y can be constructed.

Theorem (R.S., lower bound)

There is a set of cardinals $C \subseteq \{\kappa \mid \omega \leqslant \kappa \leqslant |A|^W\}$ and a cardinal χ such that, for all $\lambda \geqslant \chi$, $\langle \lambda, \lambda^+ \rangle \in \mathrm{Spec}_{\aleph}(M)$ if and only if $\mathrm{cf}(\lambda) \in C$.

Open questions

- Can we construct M such that

$$\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc} \cup \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) = \omega\} \cup \{\langle \omega, \omega \rangle\}?$$

Open questions

- Can we construct M such that

$$\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc} \cup \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) = \omega\} \cup \{\langle \omega, \omega \rangle\}?$$

Cohen's first model is almost an example, but there is no X such that $\aleph^*(X) = \omega$.

Open questions

- Can we construct M such that

$$\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc} \cup \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) = \omega\} \cup \{\langle \omega, \omega \rangle\}?$$

Cohen's first model is almost an example, but there is no X such that $\aleph^*(X) = \omega$.

- Can we have a 'break' in the oblate cardinals? That is, for some χ,

$$\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc} \cup \{\langle \lambda, \lambda^+ \rangle \mid \lambda \geqslant \chi, \mathrm{cf}(\lambda) = \omega\} \cup \{\langle \omega, \omega \rangle\}?$$

Open questions

- Can we construct M such that

$$\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc} \cup \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) = \omega\} \cup \{\langle \omega, \omega \rangle\}?$$

Cohen's first model is almost an example, but there is no X such that $\aleph^*(X) = \omega$.

- Can we have a 'break' in the oblate cardinals? That is, for some χ,

$$\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc} \cup \{\langle \lambda, \lambda^+ \rangle \mid \lambda \geqslant \chi, \mathrm{cf}(\lambda) = \omega\} \cup \{\langle \omega, \omega \rangle\}?$$

- If $\aleph(X) = \omega$ and $\aleph^*(X) = \kappa^+$, where κ is weakly inaccessible, is there Y such that $\aleph(Y) = \kappa$?

Bibliography

[Bla79] Andreas Blass. “Injectivity, projectivity, and the axiom of choice”. In: *Trans. Amer. Math. Soc.* 255 (1979), pp. 31–59. ISSN: 0002-9947,1088-6850.

[RS24] Calliope Ryan-Smith. “The Hartogs–Lindenbaum spectrum of symmetric extensions”. In: *Math. Log. Quart.* 70.2 (2024), pp. 210–223.

[Usu21] Toshimichi Usuba. “Choiceless Löwenheim-Skolem property and uniform definability of grounds”. In: *Advances in mathematical logic*. Vol. 369. Springer Proc. Math. Stat. Springer, Singapore, [2021] ©2021, pp. 161–179. ISBN: 978-981-16-4172-5; 978-981-16-4173-2.

academic.calliope.mx c.Ryan-Smith@leeds.ac.uk