

The Axiom of Extendable Choice

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The Partition Principle

Notation (Injections and surjections)

Given sets X and Y , write: $|X| \leq |Y|$ if there is an injection $X \rightarrow Y$;
 $|X| \leq^* |Y|$ if there is a surjection $Y \rightarrow X$, or if $X = \emptyset$; and $|X| = |Y|$ if there is a bijection $X \rightarrow Y$.

Theorem

For all X and Y , $|X| \leq |Y|$ if and only if $|X| \leq^ |Y|$.*

Sketch proof.

Suppose that $f: Y \rightarrow X$ is a surjection. Let $g: X \rightarrow Y$ be such that $g(x) \in f^{-1}(x)$ for all $x \in X$. Then g is an injection. □

The Partition Principle

Remark

The prior proof relies on AC, and consistently the theorem fails in ZF.

Definition (Partition Principle)

The *partition principle*, PP, is the statement “for all X and Y , $|X| \leq |Y|$ if and only if $|X| \leq^* |Y|$.”

Question: Partition Problem

Does PP imply AC?

The Cantor–Bernstein theorem

Theorem (Cantor–Bernstein theorem)

If $|X| \leq |Y|$ and $|Y| \leq |X|$, then $|X| = |Y|$.

Remarkably, the theorem is true in ZF, even without assuming AC. Though Cantor originally published the result without proof [Can87], he later produced it as an immediate corollary of the linear ordering of cardinal numbers:

Only later, when we better understand transfinite cardinal numbers, will the truth become clear: for all sets X and Y , either $|X| \leq |Y|$ or $|X| \geq |Y|$. [Can32, p. 285]

The truth becomes clear

Only later ... will the truth become clear: for all X and Y , $|X| \leq |Y|$ or $|X| \geq |Y|$.

Theorem (Hartogs [Har15])

For all X there is an ordinal α such that $|\alpha| \not\leq |X|$.

Corollary (Hartogs, *ibid.*)

AC is equivalent to the statement that, for all X and Y , $|X| \leq |Y|$ or $|X| \geq |Y|$.

Proof.

Let X be a set, and α an ordinal such that $|\alpha| \not\leq |X|$. Then $|X| \leq |\alpha|$. □

Hartogs and Lindenbaum numbers

Definition (Hartogs and Lindenbaum numbers)

The *Hartogs number* of a set X , denoted $\aleph(X)$, is the least ordinal α such that $|\alpha| \not\leq |X|$.

The *Lindenbaum number* of a set X , denoted $\aleph^*(X)$, is the least ordinal α such that $|\alpha| \not\leq^* |X|$.

Note that $\aleph(X)$ exists and is \aleph_α for some α .

Theorem (Lindenbaum, [LT26])

AC is equivalent to the statement that, for all X and Y , $|X| \leq^* |Y|$ or $|X| \geq^* |Y|$.

Note that $\aleph^*(X)$ exists, is \aleph_α for some α , and $\aleph(X) \leq \aleph^*(X)$.

Hartogs and Lindenbaum numbers

Remark

If PP holds, then for all X , $\aleph(X) = \aleph^*(X)$.

← " $\aleph = \aleph^*$ "

Question

Does $\aleph = \aleph^*$ imply AC? Or even PP?

Answer: No.

Notation

WO denotes the class of well-orderable sets. AC_{WO} means "for all $X \in WO$, if $\emptyset \notin X$ then X has a choice function."

Theorem (Pincus, [Pel78; Pin69] resp.)

1. $\aleph = \aleph^*$ is equivalent to AC_{WO} .
2. If ZF is consistent, then so is $ZF + AC_{WO} + \neg PP$ (hence $\neg AC$).

The ordinary partition problem

Question: Partition Problem

Does

“for all X and Y , if $|X| \leq^ |Y|$ then $|X| \leq |Y|$ ”*

imply

“for all X , if $\emptyset \notin X$, then X has a choice function”?

Theorem (Pincus)

It is the case that

“For all $X \in \text{WO}$ and Y , $|X| \leq^ |Y|$, then $|X| \leq |Y|$ ”*

implies

“for all $X \in \text{WO}$, $\emptyset \notin X$, then X has a choice function.”

Pincus's theorem

Theorem (Pincus, [Pel78])

$\aleph = \aleph^*$ is equivalent to AC_{WO} .

Proof.

(\Leftarrow). Let $\alpha \in \text{Ord}$ be such that $|\alpha| \leq^* |X|$, witnessed by $f: X \rightarrow \alpha$. By AC_{WO} , $\{f^{-1}(\{\beta\}) \mid \beta < \alpha\}$ has a choice function $g: \alpha \rightarrow X$, which is an injection. Hence, $\alpha < \aleph^*(X) \implies \alpha < \aleph(X)$. □

Theorem (Pincus, [Pel78])

$\aleph = \aleph^*$ is equivalent to AC_{WO} .

Proof.

(\implies). Assume $AC_{<\aleph_\delta}$ and let $X = \{X_\alpha \mid \alpha < \aleph_\delta\} \not\equiv \emptyset$. By assumption, for all $\alpha < \aleph_\delta$, $Y_\alpha = \prod_{\beta < \alpha} X_\beta \neq \emptyset$.

$$\kappa_\alpha = \aleph \left(\bigcup \{D_\beta \mid \beta < \alpha\} \right) \quad \text{and} \quad D_\alpha = Y_\alpha \times \kappa_\alpha.$$

Let $D = \bigcup_{\alpha < \aleph_\delta} D_\alpha$ and $\lambda = \sup\{\kappa_\alpha \mid \alpha < \aleph_\delta\}$. Then $|\lambda| \leq^* |D|$, so by assumption there is an injection $f: \lambda \rightarrow D$.

$f(0) = \langle g_0, \varepsilon_0 \rangle$, where $g_0 \in Y_\gamma = \prod_{\alpha < \gamma} X_\alpha$ some γ and $\varepsilon_0 < \kappa_\gamma$. For all $\alpha < \gamma$, let $x_\alpha = g_0(\alpha) \in X_\alpha$.

Let $x_\alpha = g_\beta(\alpha)$, where β is least such that $\alpha \in \text{dom}(g_\beta)$.

If this is well-defined then we're done. Otherwise, $f''\lambda \subseteq \bigcup_{\beta < \alpha} D_\beta$ for some β . However, $\lambda \geq \aleph(\bigcup_{\beta < \alpha} D_\beta)$ by construction. □

A Pincus-style characterisation

Theorem (Pincus, [Pel78])

$\aleph = \aleph^*$ is equivalent to AC_{WO} .

Definition (Hartogs–Lindenbaum spectrum)

Given a model $M \models ZF$, the *Hartogs–Lindenbaum spectrum* of M is the class

$$\text{Spec}_{\aleph}(M) := \{\langle \aleph(X), \aleph^*(X) \rangle \mid X \in M\}.$$

Note that $\text{Suc} := \{\langle \aleph_{\alpha+1}, \aleph_{\alpha+1} \rangle \mid \alpha \in \text{Ord}\} \subseteq \text{Spec}_{\aleph}(M)$.

Theorem (R.S., [Rya24])

AC_{WO} is equivalent to $\text{Suc} = \text{Spec}_{\aleph}(M)$.

Theorem (Karagila–R.S., [KR24])

There is a model M of ZF such that

$$\text{Spec}_{\aleph}(M) = \{\langle \lambda, \kappa \rangle \mid \aleph_0 \leq \lambda \leq \kappa\}.$$

That is,

$$M \models (\forall \lambda \leq \kappa \text{ infinite cardinals})(\exists X)\aleph(X) = \lambda \wedge \aleph^*(X) = \kappa.$$

Failures of AC_{ω_0}

Cohen's first model

Cohen's first model, M , is a model of ZF such that $L \subseteq M \subseteq L[(a_i)_{i < \omega}]$, where each a_i is a Cohen real. In fact, $M = L(A)$, where $A = \{a_i \mid i < \omega\}$.

Facts

1. $M \models \aleph(A) = \aleph_0$.
2. ZF proves that if $X \subseteq \mathbb{R}$ is infinite, then $|\aleph_0| \leq^* |X|$.
3. So $\aleph(A) = \aleph_0 < \aleph^*(A) = \aleph_1$.

Theorem (R.S., [Rya24])

Let M be Cohen's first model. Then

$$\text{Spec}_{\aleph}(M) = \text{Suc} \cup \{\langle \lambda, \lambda^+ \rangle \mid \text{cf}(\lambda) = \omega\}.$$

The Axiom of Extendable Choice

Definition (Extendable Choice)

Let α be an ordinal. An (indexed) family $X = \{X_\beta \mid \beta < \alpha\}$ of sets is α -*approachable* if for all $\beta < \alpha$: $X_\beta \neq \emptyset$ and $\{X_\gamma \mid \gamma < \beta\}$ has a choice function.

The *axiom of extendable choice* (for α), written EC_α , is the statement that every α -approachable family has a choice function.

EC_α was originally introduced as an unnamed axiom by Levy [Lév64]¹ with the notation $C(\alpha)$. 'EC _{α} ' is from [Rya25].

¹In which he proves a lot about EC and its relationship with other choice principles.

The Axiom of Extendable Choice

Theorem (Levy [Lév64])

If $\text{cf}(\alpha) = \text{cf}(\beta)$ then $\text{EC}_\alpha \longleftrightarrow \text{EC}_\beta$.

Theorem (R.S. [Rya25])

Let κ be a regular cardinal and $\text{CF}(\kappa) = \{\lambda \in \text{Card} \mid \text{cf}(\lambda) = \kappa\}$. The following are equivalent:

1. $\neg \text{EC}_\kappa$;
2. there is a limit $\mu \in \text{CF}(\kappa)$ and a set X such that $\aleph(X) = \mu$;
3. there is $\mu \in \text{CF}(\kappa)$ and a set X such that $\aleph(X) = \mu < \aleph^*(X)$; and
4. there is $\mu^* \in \text{CF}(\kappa)$ such that for all $\mu \in \text{CF}(\kappa)$ with $\mu > \mu^*$, there is X with $\aleph(X) = \mu < \aleph^*(X)$.

Failures of AC_{ω_0}

Theorem (R.S. [Rya24])

Let M be Cohen's first model. Then

$$\text{Spec}_{\aleph}(M) = \text{Suc} \cup \{ \langle \lambda, \lambda^+ \rangle \mid \text{cf}(\lambda) = \omega \}.$$

Thus, $M \models (\forall \alpha) \text{EC}_{\alpha} \longleftrightarrow \text{cf}(\alpha) \neq \omega$.

Theorem A (R.S. [Rya25])

Let $V \models \text{GCH}$ and let \mathcal{C} be a class of regular cardinals closed under regular limits. Then there is a model $M \supseteq V$ of ZF such that

$$\text{Spec}_{\aleph}(M) = \text{Suc} \cup \{ \langle \lambda, \lambda^+ \rangle \mid \text{cf}(\lambda) \in \mathcal{C} \}.$$

Thus, $M \models (\forall \alpha) \text{EC}_{\alpha} \longleftrightarrow \text{cf}(\alpha) \notin \mathcal{C}$.

Different cardinal structure

Let \mathfrak{C} represent the class of all cardinal numbers.

Theorem (Hartogs [Har15] and Lindenbaum [LT26] resp.)

1. AC is equivalent to " $\langle \mathfrak{C}, \leq \rangle$ is a linear order."
2. AC is equivalent to " $\langle \mathfrak{C}, \leq^* \rangle$ is a linear order."

Theorem (Jech [Jec66], Karagila [Kar14], Shen-Zhou [SZ25])

1. For all partial orders $\langle P, \leq \rangle$, there is a model of ZF such that P embeds into $\langle \mathfrak{C}, \leq \rangle$ (Jech) and $\langle \mathfrak{C}, \leq^* \rangle$ (Karagila).
2. For all doubly ordered sets $\langle P, \leq, \leq^* \rangle$,^a there is a model of ZF such that P embeds into $\langle \mathfrak{C}, \leq, \leq^* \rangle$ (Shen-Zhou).

^a $\langle P, \leq \rangle$ is a partial order, $\langle P, \leq^* \rangle$ is a preorder and $\leq \subseteq \leq^*$.

Future work

1. Finer control over the Hartogs–Lindenbaum spectrum (e.g. removing the assumptions from Theorem A).
2. Embedding more general structures into the cardinals (e.g. cardinal algebras).
3. Understanding how EC_κ is affected by large cardinal properties of κ .
4. Localising results to SVC.

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