The Hartogs–Lindenbaum Spectrum

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How big is a set?

Definition

Let X, Y be sets.

- $|X| \leqslant |Y|$ if there is an injection $X \to Y$.
- $|X| \leqslant^* |Y|$ if there is a surjection $Y \to X$, or $X = \emptyset$.
- $|X| = |Y|$ if there is a bijection $X \to Y$.

Theorem (Zermelo, *Well-ordering principle*)

(ZFC) For all X there is an ordinal α such that $|X| = |\alpha|$.

So the size of X is $|X| = \min\{\alpha \in \mathrm{Ord} \mid |X| = |\alpha|\}$.

(Some) consequences of the Well-ordering Principle

1. $|X| \leqslant |Y|$ if and only if $|X| \leqslant^* |Y|$.
2. If X is infinite, then $|\omega| \leqslant |X|$ (and $|\omega| \leqslant^* |X|$).
3. If X is infinite, then $|X \cup \{X\}| = |X|$.

Non-example (Cohen's first model)

In Cohen's first model of ZF $+ \neg$AC there is an infinite set A such that there is no injection $\omega \to A$. This violates Consequence (2)![a] However, there *is* a surjection $A \to \omega$, also violating Consequence (1).

[a] This also violates Consequence (3) (exercise).

The definition $|A| = \min\{\alpha \in \mathrm{Ord} \mid |A| = |\alpha|\}$ is invalid.

Approximate size

Definition (Hartogs and Lindenbaum numbers)

Let X be a set. The **Hartogs number** of X, denoted $\aleph(X)$, is

$$\min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant |X|\}.$$

The **Lindenbaum number** of X, denoted $\aleph^*(X)$, is

$$\min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant^* |X|\}.$$

Fact/Theorem (Hartogs/Lindenbaum [Har15; LT26])/Exercise

For all X:

- $\aleph(X)$ and $\aleph^*(X)$ exist.
- $\aleph(X)$ and $\aleph^*(X)$ are cardinals (i.e. $|\alpha| < |\aleph(X)|$ for all $\alpha < \aleph(X)$).
- $\aleph(X) \leqslant \aleph^*(X)$.

$$\aleph(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq |X|\}, \qquad \aleph^*(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq^* |X|\}$$

Example

If X can be well-ordered, say $|X| = \lambda$, then $\aleph(X) = \aleph^*(X) = \lambda^+$.

In Cohen's first model, $\aleph(A) = \omega$, and $\aleph^*(A) = \omega_1$.

Definition

A set X is **eccentric** if $\aleph(X) < \aleph^*(X)$.

So if $V \vDash \mathrm{ZFC}$ then there are *no* eccentric sets: $\aleph(X) = \aleph^*(X) = |X|^+$.

Question

What are the eccentric sets in Cohen's first model?

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Theory. Maybe all eccentric sets look like A: If X is eccentric then $\aleph(X) = \omega$ and $\aleph^*(X) = \omega_1$.

Answer. No.

$\aleph^*(A) = \omega_1$, so there is a partition $A = \bigsqcup_{n\in\omega} A_n$ such that $A_n \neq \emptyset$ for all n. Let

$$B_m = \bigcup_{n<m} A_n \times \omega_n, \qquad B = \bigcup_{n\in\omega} A_n \times \omega_n.$$

Lemma

$\aleph(B) = \omega_\omega$ *and* $\aleph^*(B) = \omega_{\omega+1}$.

$$B_m = \bigcup_{n<m} A_n \times \omega_n, \qquad B = \bigcup_{n\in\omega} A_n \times \omega_n.$$

Lemma

$\aleph(B) = \omega_\omega$ *and* $\aleph^*(B) = \omega_{\omega+1}$.

Proof.

($\aleph(B) = \omega_\omega$). For all $n \in \omega$, $\omega_n \hookrightarrow A_n \times \omega_n \hookrightarrow B$, so $\aleph(B) \geqslant \omega_\omega$.

Fact/exercise: For all $m \in \omega$, $\aleph(B_m) = \omega_m$.

So if $f\colon \omega_\omega \hookrightarrow B$, then $f``\omega_\omega \not\subseteq B_m$ for all m. Then f lets us define an injection $\omega \to A$, a contradiction.

($\aleph^*(B) = \omega_{\omega+1}$). Projection is a surjection $B \to \omega_\omega$, so $\aleph^*(B) \geqslant \omega_{\omega+1}$.

$B \subseteq A \times \omega_\omega$, so $\aleph^*(B) \leqslant \aleph^*(A \times \omega_\omega) \leqslant \aleph^*(A)^+ \times \omega_{\omega+1} = \omega_{\omega+1}$.[a] □

[a] For all X, $\aleph^*(A \times \lambda) \leqslant \max\{\aleph^*(A)^+, \lambda^+\}$.

Looking at the proof, if $\mathrm{cf}(\lambda) = \omega$ then the same construction works with ω_ω replaced by λ.

Lemma (R.S. [RS24])

(Cohen's first model) If $\mathrm{cf}(\lambda) = \omega$ then there is X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \lambda^+$.

This is *all* of the eccentric sets.

Fact

(Cohen's first model). For all X, either:

- *$\aleph(X) = \aleph^*(X) = \kappa^+$ some κ, or*
- *$\aleph(X) = \lambda$, $\aleph^*(X) = \lambda^+$, and $\mathrm{cf}(\lambda) = \omega$.*

Definition (Hartogs–Lindenbaum spectrum)

Let $M \vDash$ ZF. The **Hartogs–Lindenbaum spectrum** of M is

$$\mathrm{Spec}_{\aleph}(M) := \{\langle \aleph(X), \aleph^*(X)\rangle \mid X \in M\},$$

the class of pairs $\langle \lambda, \kappa \rangle$ such that $(\exists X \in M) M \vDash \aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$.

For all well-ordered cardinals λ, $\aleph(\lambda) = \aleph^*(\lambda) = \lambda^+$, so for all $M \vDash$ ZF,

$$\mathrm{Suc} := \left\{\langle \lambda^+, \lambda^+ \rangle \;\middle|\; \lambda \text{ a cardinal}\right\} \subseteq \mathrm{Spec}_{\aleph}(M).$$

Example

- If $M \vDash$ ZFC then $\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc}$.
- If M is Cohen's first model, $\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc} \cup \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) = \omega\}$.
- [KRS23] There is $M \vDash$ ZF such that $\mathrm{Spec}_{\aleph}(M) = \{\langle n+1, n+1 \rangle \mid n \in \omega\} \cup \{\langle \lambda, \kappa \rangle \mid \lambda \leqslant \kappa \text{ are infinite}\}$.

Theorem (R.S. [RS24])

If there is A such that $\aleph(A) \leqslant \mu < \aleph^*(A)$, where μ is a regular cardinal, then for all λ large enough such that $\mathrm{cf}(\lambda) = \mu$, there is a set B such that $\aleph(B) = \lambda$ and $\aleph^*(B) = \lambda^+$.

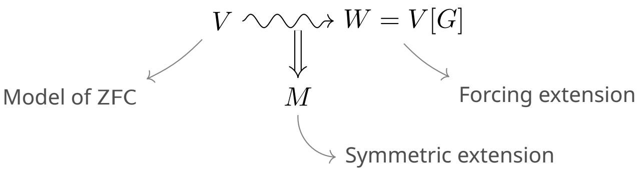
The proof is the same as in Cohen's first model: If $A = \bigsqcup_{\alpha\in\mu} A_\alpha$ and $\lambda = \sup\{\lambda_\alpha \mid \alpha \in \mu\}$, then let $B = \bigcup_{\alpha\in\mu} A_\alpha \times \lambda_\alpha$.

Corollary

*Either there are **no** eccentric sets, or there is a **proper class** of eccentric sets.*

Symmetric extensions

Instead of considering *all* models of ZF, let us consider only *symmetric extensions*:



Many, many models of ZF $+ \neg$AC are obtained as symmetric extensions.

SVC

Definition

For $M \vDash$ ZF and $A \in M$, SVC(A) is the statement

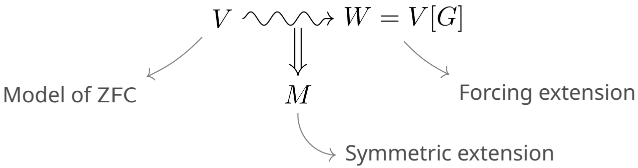
$$(\forall X)(\exists \eta \in \mathrm{Ord})\,|X| \leqslant^* |A \times \eta|\,.$$

That is, for all $X \in M$ there is an ordinal η such that there is a surjection $f\colon A \times \eta \to X$ in M.

We write SVC, **small violations of choice**, to mean $(\exists A)$SVC(A).

Theorem (Blass/Usuba [Bla79; Usu21])

M is a symmetric extension of a model of ZFC *if and only if* $M \vDash$ SVC.



We will assume that M and W agree on cardinality and cofinality of ordinals.

Example

Cohen's first model is a model of SVC(A), where A is our eccentric set from earlier. Also, $W \vDash |A| = \omega$.[a]

[a] This is why $\aleph^*(A) < \omega_2$! If $M \vDash |\omega_1| \leqslant^* |A|$, then $W \vDash |\omega_1| \leqslant^* |A| = |\omega|$.

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$, *then* $\mathrm{cf}(\mu) \leqslant |A|^W$.

Proof.

Let $\eta \in \mathrm{Ord}$ be such that $|X| \leqslant^* |A \times \eta|$, say $g \colon A \times \eta \twoheadrightarrow X$.

For $a \in A$, let $X_a = g``(\{a\} \times \eta)$. Note that X_a is well-orderable in M.

Let $\mu < \aleph^*(X)$, so there is a surjection $f \colon X \twoheadrightarrow \mu$.

In W, $\mu = \bigcup_{a \in A} f``X_a$. If $\mathrm{cf}(\mu) > |A|^W$, then there is $a \in A$ such that $|f``X_a| = |\mu|$.

Since $|f``X_a| = |\mu|$ we get $|\mu| \leqslant^* |X_a|$ (in M).

X_a is well-orderable, so $|\mu| \leqslant^* |X_a| \implies |\mu| \leqslant |X_a|$.

Hence $|\mu| \leqslant |X|$, and $\mu < \aleph(X)$. □

Theorem (R.S. [RS24])

If $M \vDash \mathsf{SVC}(A)$ *and* $\aleph(X) \leqslant \mu < \aleph^*(X)$*, then* $\mathrm{cf}(\mu) \leqslant |A|^W$*.*

Corollary

If $|A|^W \leqslant \aleph(X) < \aleph^*(X)$ *then* $\mathrm{cf}(\aleph(X)) \leqslant |A|^W$ *and* $\aleph^*(X) = \aleph(X)^+$*.*

Proof.

Let $\lambda = \aleph(X)$. Then $\aleph(X) \leqslant \lambda < \aleph^*(X)$, so $\mathrm{cf}(\lambda) \leqslant |A|^W$.
If $\lambda^+ < \aleph^*(X)$ then $\aleph(X) \leqslant \lambda^+ < \aleph^*(X)$, so $\mathrm{cf}(\lambda^+) \leqslant |A|^W$.
However, $\mathrm{cf}(\lambda^+) = \lambda^+ > |A|^W$, contradicting that $\lambda^+ < \aleph^*(X)$. □

This is why $\mathrm{Spec}_\aleph(M) = \mathrm{Suc} \cup \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) = \omega\}$: If $\aleph(X) < \aleph^*(X)$ then $\mathrm{cf}(\aleph(X)) = \omega$ and $\aleph^*(X) = \aleph(X)^+$.

Let $M \vDash \text{SVC}(A)$. Then:

1. $\mathrm{Suc} = \{\langle\lambda^+, \lambda^+\rangle \mid \lambda \text{ a cardinal}\} \subseteq \mathrm{Spec}_\aleph(M)$.
2. If $\aleph(X)^+ < \aleph^*(X)$ then $\aleph(X) < |A|^W$.
3. $C = \{\mu \mid (\exists X)\,\mathrm{cf}(\aleph(X)) = \mu \wedge \aleph(X) < \aleph^*(X)\} \subseteq \{\mu \mid \mu \leqslant |A|^W\}$ is a **set**.
4. If $\mu \in C$ then for **all large enough** λ such that $\mathrm{cf}(\lambda) = \mu$ there is X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \lambda^+$.
   So there is χ such that, for all $\lambda \geqslant \chi$,

   $(\exists X)\lambda = \aleph(X) < \aleph^*(X)$ **if and only if** $\mathrm{cf}(\lambda) \in C$ and $\aleph^*(X) = \lambda^+$.

$$\mathrm{Spec}_\aleph(M) = \bigcup \begin{cases} \mathrm{Suc} = \{\langle\lambda^+, \lambda^+\rangle \mid \lambda \in \mathrm{Card}\} \\ \mathfrak{D} \subseteq \{\langle\lambda, \kappa\rangle \mid \lambda \leqslant \kappa,\ \lambda < |A|^W\} \\ \mathfrak{C} \subseteq \{\langle\lambda, \lambda^+\rangle \mid \mathrm{cf}(\lambda) \in C, \lambda < \chi\} \\ \mathfrak{U} = \{\langle\lambda, \lambda^+\rangle \mid \mathrm{cf}(\lambda) \in C, \lambda \geqslant \chi\}. \end{cases}$$

Open questions

1. How much of a gap can we have in the spectrum? E.g., is ZF consistent with

$$\mathrm{Spec}_{\aleph}(M) = \mathrm{Suc} \cup \{\langle \omega_1, \omega_2 \rangle\} \cup \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) = \omega_1,\ \lambda \geqslant \kappa\}$$

   for arbitrarily large κ?

2. Lindenbaum numbers are not quite multiplicative:

$$\aleph^*(A) \times \aleph^*(B) \leqslant \aleph^*(A \times B) \leqslant (\aleph^*(A) \times \aleph^*(B))^+.$$

   When is either bound reached? When is either bound *possible* to reach?

3. If $M \vDash$ SVC, when can AC be forced without collapsing cardinals?

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