LOCAL REFLECTIONS OF CHOICE

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Abstract. Under the assumption of small violations of choice with seed S ($\mathsf{SVC}(S)$), the failure of many choice principles reflect to local properties of S, which can be a helpful characterisation for preservation proofs. We demonstrate the reflections of DC, AC_λ, PP, and other important forms of choice. As a consequence, we show that if S is infinite then S can be partitioned into ω many non-empty subsets.

1. Introduction

It is often the case that violating a consequence of choice is easier than verifying that a consequence of choice has been preserved. For example, to violate AC_ω in a symmetric extension, one need only add a countable family with no choice function. On the other hand, to ensure that AC_ω has been preserved one must check 'every' countable family. Blass's *small violations of choice* afford us an alternative approach. In any symmetric extension of a model of ZFC there is a 'seed' S such that $\mathsf{SVC}(S)$ holds: For all non-empty X there is an ordinal η such that $S \times \eta$ surjects onto X. Under this assumption, many violations of choice are reflected back and witnessed locally to S. For example, to verify AC_ω, one need only check $\mathsf{AC}_\omega(S)$. We prove local equivalents of several common forms of choice, such as the principle of dependent choices and well-ordered choice. The following summarises these results. All notation will be introduced in the text as the results are proved.

Theorem. *Assume* $\mathsf{SVC}(S)$ *and* $\mathsf{SVC}^+(T)$.

1. DC_λ *is equivalent to "every λ-closed subtree of $S^{<\lambda}$ has a maximal node or cofinal branch".*
2. AC_λ *is equivalent to* $\mathsf{AC}_\lambda(S)$*, which is in turn equivalent to "every function $g\colon S \to \lambda$ splits".*
3. AC_X *is equivalent to* $\mathsf{AC}_X(S)$.
4. $\mathsf{AC}_{\mathsf{WO}}$ *is equivalent to* $\mathsf{AC}_{<\aleph^*(S)}(S)$.
5. *Assume that* $\mathrm{cf}(\omega_1) = \omega_1$*. Then* CUT *is equivalent to* $\mathsf{CUT}(T)$.
6. *(Blass, [1])* AC *is equivalent to "S can be well-ordered".*
7. W_λ *is equivalent to* $\mathsf{W}_\lambda(T)$.
8. W^*_λ *is equivalent to* $\mathsf{W}^*_\lambda(T)$.
9. *(Pincus, [1])* BPI *is equivalent to "there is a fine ultrafilter on $[S]^{<\omega}$".*
10. *(Karagila–Schilhan [6])* KWP^*_α *is equivalent to "there is $\eta \in \mathrm{Ord}$ such that $|S| \leq^* |\mathscr{P}^\alpha(\eta)|$".*
11. *(Karagila–Schilhan [6])* KWP_α *is equivalent to "there is $\eta \in \mathrm{Ord}$ such that $|T| \leq |\mathscr{P}^\alpha(\eta)|$".*

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12. PP *is equivalent to* $\mathsf{PP}\restriction T \wedge \mathsf{AC_{WO}}$.
13. $\mathsf{PP}(S)\wedge\mathsf{AC_{WO}}$ *implies* $\mathsf{SVC}^+(S)$. *Hence,* PP *is equivalent to* $\mathsf{PP}(S)\wedge\mathsf{PP}\restriction S\wedge \mathsf{AC_{WO}}$.

1.1. **Structure of the paper.** In Section 2 we go over some preliminaries for the paper. In particular, in Section 2.1 we introduce the concept of a *minitial* function that will be helpful for some proofs. In Section 3 we describe reflections of various consequences of choice in the context of small violations of choice.

2. Preliminaries

We work in ZF. We denote the class of ordinals by Ord. Given a set of ordinals X, we use $\mathrm{ot}(X)$ to denote the order type of X. For sets X, Y, $|X| \leq |Y|$ means that there is an injection $X \to Y$, $|X| \leq^* |Y|$ means that there is a partial surjection $Y \to X$, and $|X| = |Y|$ means that there is a bijection $X \to Y$. For a well-orderable set X, we use $|X|$ to mean $\min\{\alpha \in \mathrm{Ord} \mid |\alpha| = |X|\}$. By a *cardinal* we mean a well-ordered cardinal, that is $\alpha \in \mathrm{Ord}$ such that $|\alpha| = \alpha$. For a set Y and a cardinal κ, we write $[Y]^\kappa$ to mean $\{A \subseteq Y \mid |A| = |\kappa|\}$ and $[Y]^{<\kappa}$ to mean $\{A \subseteq Y \mid |A| < |\kappa|\}$. Given a set X, the *Lindenbaum number* of X is $\aleph^*(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq^* |X|\}$. It is a theorem of ZF that $\aleph^*(X)$ is well-defined and a cardinal. We denote concatenation of tuples by $^\frown$.

Small violations of choice was introduced in [1]. For a set S (known as the *seed*), $\mathsf{SVC}(S)$ is the statement "for all X there is an ordinal η such that $|X| \leq^* |S\times\eta|$", and SVC is the statement $(\exists S)\mathsf{SVC}(S)$. We shall also make use of the injective form, $\mathsf{SVC}^+(S)$ meaning "for all X, there is an ordinal η such that $|X| \leq |S\times\eta|$". See [7] for a more detailed overview of SVC and SVC^+.

Fact. $\mathsf{SVC}^+(S) \implies \mathsf{SVC}(S) \implies \mathsf{SVC}^+(\mathscr{P}(S))$.

2.1. **Minitial functions.** Let η be an ordinal and $f\colon S\times\eta \to X$ be a surjection. Let D be the set $\{\langle s,\alpha\rangle \mid (\forall\beta < \alpha) f(s,\beta) \neq f(s,\alpha)\}$. Then $f\restriction D$ is still a surjection and, for all $s\in S$, $f\restriction(D\cap\{s\}\times\eta)$ is an injection. For $s \in S$, let $\eta_s = \mathrm{ot}(\{\alpha<\eta \mid \langle s,\alpha\rangle \in D\})$. Then there is a partial surjection $f'\colon S\times\eta\to X$ such that $\mathrm{dom}(f) = \bigcup_{s\in S}\{s\}\times\eta_s$ given by setting $f'(s,\alpha)$ to be the αth element of $f``\{s\}\times\eta$.

Definition 2.1. A partial surjection $f\colon S\times\eta\to X$ is *minitial* if:

- η is least among ordinals such that $|X| \leq^* |S\times\eta|$;
- for all $s\in S$ there is $\eta_s \leq \eta$ such that $\mathrm{dom}(f)\cap\{s\}\times\eta = \{s\}\times\eta_s$; and
- for all $s\in S$, $f\restriction\{s\}\times\eta_s$ is an injection.

An injection $f\colon X\to S\times\eta$ is a *minitial embedding* if:

- η is least among ordinals such that $|X| \leq |S\times\eta|$; and
- for all $s\in S$, there is $\eta_s\leq\eta$ such that $f``X\cap\{s\}\times\eta = \{s\}\times\eta_s$.

A set $X\subseteq S\times\eta$ is *minitial* if the inclusion $X\hookrightarrow S\times\eta$ is a minitial embedding.

Our remarks beforehand show that if η is least such that $|X| \leq^* |S\times\eta|$ then there is a minitial partial surjection $S\times\eta\to X$. Similarly, if η is least such that $|X|\leq|S\times\eta|$ then there is a minitial embedding $X\to S\times\eta$.

2.2. **Forcing and symmetric extensions.** While no knowledge of forcing or symmetric extensions are required for the main results, it is used in the proof of Proposition 3.18. For an introduction to forcing and symmetric extensions one can go to [4, Chapters 14 and 15], while for an overview of the notation and terminology used in the proof of Proposition 3.18, one should see [5].

3. Reflections

3.1. **The principle of dependent choices.** A *tree* is a partially ordered set $\langle T, \leq\rangle$ with minimum element 0_T such that, for all $t \in T$, $\{s \in T \mid s \leq t\}$ is well-ordered by $\leq$. This gives rise to a notion of rank $\mathrm{rk}(t) = \sup\{\mathrm{rk}(s) + 1 \mid s <_T t\}$, of height $\mathrm{ht}(T) = \sup\{\mathrm{rk}(t) + 1 \mid t \in T\}$, and of levels $T_\alpha = \{t \in T \mid \mathrm{rk}(t) = \alpha\}$. For an ordinal α, T is *α-closed* if every chain in T of order type less than α has an upper bound. A *branch* is a maximal chain $b \subseteq T$, and b is *cofinal* if for all $\alpha < \mathrm{ht}(T)$, $b \cap T_\alpha \neq \emptyset$. For a cardinal λ, DC_λ is the statement "every λ-closed tree has a maximal node or a cofinal branch". Note that if DC_λ fails then this must be witnessed by a tree of height at most λ.

Proposition 3.1. *Assume* $\mathsf{SVC}(S)$. *The following are equivalent:*

1. DC_λ*;*
2. *every λ-closed subtree of $S^{<\lambda}$ has a maximal node or cofinal branch.*

Proof. Certainly if DC_λ holds then DC_λ holds for subtrees of $S^{<\lambda}$, so assume instead that every λ-closed subtree of $S^{<\lambda}$ has a maximal node or cofinal branch. Let T be a λ-closed tree of height $\delta \leq \lambda$ with no maximal nodes. Let $\eta \in \mathrm{Ord}$ and $f\colon S \times \eta \to T$ be a surjection. We shall define $A \subseteq S^{<\lambda}$ and a tree embedding $\iota\colon A \to T$ by induction on the levels of T. Firstly, let $A_0 = \{\langle\rangle\}$ and $\iota(\langle\rangle) = 0_T$. Suppose that $A_\alpha \subseteq S^{\leq\alpha}$ and $\iota \restriction A_\alpha$ have been defined. Let

$$A_{\alpha+1} = \{x^\frown\langle s\rangle \in A_\alpha \times S \mid (\exists\gamma < \eta)\iota(x) <_T f(s,\gamma) \wedge f(s,\gamma) \in T_{\alpha+1}\}$$

and let $\iota(x^\frown\langle s\rangle) = f(s, \gamma^*)$, where γ^* is least among all possible γ. Since T has no maximal nodes, $A_{\alpha+1} \neq \emptyset$ as long as $A_\alpha \neq \emptyset$, and every $x \in A_\alpha$ extends to at least one $x^\frown\langle s\rangle$ in $A_{\alpha+1}$. At limit stages $\alpha < \delta$ let A_α be the set of all cofinal branches through $\bigcup_{\beta<\alpha} A_\beta$. If there are none, then $\bigcup_{\beta<\alpha} A_\beta$ is λ-closed (by taking unions) with no maximal nodes or cofinal branches, contradicting our assumption. Given a cofinal branch x of $\bigcup_{\beta<\alpha} A_\beta$, by the λ-closure of T, $\iota(x) = \sup\{\iota(x \restriction \beta) \mid \beta < \alpha\} \in T$ is well-defined. Let $A = \bigcup_{\alpha<\delta} A_\alpha$. If $b \subseteq A$ is a cofinal branch, then $\mathrm{ot}(b) = \delta$ and $\iota``b$ is a chain of order type δ in T, and hence is cofinal. Thus, DC_λ holds. $\square$

3.2. **Well-ordered choice.** For a set X, AC_X is the statement "if $\emptyset \neq Y$, $\emptyset \notin Y$, and $|Y| \leq^* |X|$, then $\prod Y \neq \emptyset$", where $\prod Y = \{f\colon Y \to \bigcup Y \mid (\forall y \in Y) f(y) \in y\}$ is the set of choice functions. $\mathsf{AC}_{\mathsf{WO}}$ means $(\forall\alpha \in \mathrm{Ord})\mathsf{AC}_\alpha$. $\mathsf{AC}_X(A)$ is AC_X for families of subsets of A: If $\emptyset \neq Y \subseteq \mathscr{P}(A) \setminus \{\emptyset\}$ and $|Y| \leq^* |X|$ then $\prod Y \neq \emptyset$.

Given a function $g\colon X \to Y$, we say that g *splits* if there is a partial function $h\colon Y \to X$ such that $\mathrm{dom}(h) = g``X$ and $gh(y) = y$ for all $y \in g``X$.

Proposition 3.2. *Assume* $\mathsf{SVC}(S)$. *The following are equivalent:*

1. AC_λ*;*
2. $\mathsf{AC}_\lambda(S)$*;*
3. *every function $g\colon S \to \lambda$ splits.*

Proof. Certainly AC_λ implies $\mathsf{AC}_\lambda(S)$. Assuming $\mathsf{AC}_\lambda(S)$, if $g\colon S \to \lambda$ then we define

$$Y = \left\{g^{-1}(\{\alpha\}) \;\middle|\; \alpha \in g``S\right\}.$$

Then if $c \in \prod Y$, $h\colon g``S \to S$ given by $h(\alpha) = c(g^{-1}(\alpha))$ splits g.

Finally, assume that every $g\colon S \to \lambda$ splits and let $X = \{X_\alpha \mid \alpha < \lambda\}$ be a collection of non-empty sets. Let $f\colon S \times \eta \to \bigcup X$ be a surjection for some $\eta \in \mathrm{Ord}$. For $\alpha < \lambda$, let $\beta_\alpha = \min\{\beta < \eta \mid (f``S \times \{\beta\}) \cap X_\alpha \neq \emptyset\}$, and let $S_\alpha = \{s \in S \mid f(s, \beta_\alpha) \in X_\alpha\}$. Let $g(s) = \min\{\alpha < \lambda \mid s \in S_\alpha\}$. If $h\colon \lambda \to S$ is a partial function splitting g, then setting $\gamma_\alpha = \min\{\gamma < \lambda \mid h(\gamma) \in X_\alpha\}$ is well-defined, and $C(\alpha) = f(h(\gamma_\alpha), \beta_\alpha)$ is a choice function for X. $\square$

Corollary 3.3. *Assume* $\mathsf{SVC}(S)$*. The following are equivalent:*

1. $\mathsf{AC}_{\mathsf{WO}}$*;*
2. $\mathsf{AC}_{<\aleph^*(S)}(S)$.

Proof. Certainly $\mathsf{AC}_{\mathsf{WO}}$ implies $\mathsf{AC}_{<\aleph^*(S)}(S)$, so assume $\mathsf{AC}_{<\aleph^*(S)}(S)$. Let $g\colon S \to \lambda$, $A = g``S$, and $\alpha = \mathrm{ot}(A) < \aleph^*(S)$. Taking $\iota\colon A \to \alpha$ to be the unique isomorphism, we have $\iota \circ g\colon S \to \alpha$. By $\mathsf{AC}_{<\aleph^*(S)}(S)$, $\iota \circ g$ is split, say by $f\colon \alpha \to S$. Then $f \circ \iota^{-1}$ splits g. Since g was arbitrary, Proposition 3.2 gives us AC_λ. Since λ was arbitrary, we obtain $\mathsf{AC}_{\mathsf{WO}}$ as required. □

Corollary 3.4. *Assume* $\mathsf{SVC}(S)$*. Then* $\aleph^*(S) \neq \aleph_0$*.*

Proof. If $\aleph^*(S) = \aleph_0$ then there are no surjections $S \to \omega$, so every function $S \to \omega$ has finite image and hence splits. Therefore, AC_ω holds. However, AC_ω implies that for all X, $\aleph^*(X) \neq \aleph_0$. □

We also have the following more direct (albeit longer) proof of this corollary.

Alternative proof. Assume $\mathsf{SVC}(S)$, where S is infinite (the case of S finite immediately gives $\aleph^*(S) \neq \aleph_0$), and let $f\colon S \times \eta \to S^{\underline{<\omega}}$ be a minitial surjection, where $S^{\underline{n}}$ is the set of injections $n \to S$ and $S^{\underline{<\omega}} = \bigcup_{n<\omega} S^{\underline{n}}$. For $s \in S$, let $\eta_s = \{\alpha < \eta \mid \langle s, \alpha\rangle \in \mathrm{dom}(f)\}$. If $\{\eta_s \mid s \in S\}$ is infinite then $|\omega| \leq^* |S|$ as required. Otherwise, if there is $s \in S$ such that $\eta_s \geq \omega$ then, by minitiality, $f \upharpoonright \{s\} \times \eta_s \to S^{\underline{<\omega}}$ is an injection, and hence S is Dedekind-infinite. In particular, $|\omega| \leq^* |S|$. Finally, if η_s finite for all s, then η is finite. $\aleph^*(S \times \eta) \geq \aleph^*(S^{\underline{<\omega}}) \geq \aleph_1$, but by the additivity of Lindenbaum numbers, $\aleph^*(S \times \eta) = \aleph^*(S)$, and hence $\aleph^*(S) \geq \aleph_1$. □

The idea behind Corollary 3.4 extends to certain other cardinals κ, though we additionally have to assume $\mathsf{AC}_{<\kappa}$, as (unlike $\mathsf{AC}_{<\omega}$) it is not automatic.

Proposition 3.5. *Let* κ *be a limit cardinal or singular. Assume* $\mathsf{SVC}(S)$ *and* $\mathsf{AC}_{<\kappa}$*. Then* $\aleph^*(S) \neq \kappa$*.*

Proof. If $\aleph^*(S) = \kappa$ then, by Corollary 3.3, $\mathsf{AC}_{\mathsf{WO}}$ holds. However, by [7, Theorem 3.4], $\mathsf{AC}_{\mathsf{WO}}$ is equivalent to "for all X, $\aleph^*(X)$ is a regular successor", contradicting that $\aleph^*(S) = \kappa$ is singular or a limit. □

In fact, the method of Proposition 3.2 applies more generally.

Proposition 3.6. *Assume* $\mathsf{SVC}(S)$*. The following are equivalent:*

- AC_X*;*
- $\mathsf{AC}_X(S)$.

Proof. Certainly AC_X implies $\mathsf{AC}_X(S)$. so assume $\mathsf{AC}_X(S)$ and let $p\colon X \to Y$ be a surjection. Let $f\colon S \times \eta \to \bigcup Y$ be a minitial surjection. For $x \in X$, let

$$\beta_x = \min\{\beta < \eta \mid (f``S \times \{\beta\}) \cap p(x) \neq \emptyset\}, \text{ and}$$
$$S_x = \{s \in S \mid f(s, \beta_x) \in p(x)\}.$$

By $\mathsf{AC}_X(S)$, we have $c \in \prod\{S_x \mid x \in X\}$, giving $g(p(x)) := f(c(x), \beta_x) \in \prod Y$. □

3.3. **The countable union theorem.** For a set X, we write $\mathsf{CUT}(X)$ to mean "a countable union of countable subsets of X is countable", and CUT to mean the countable union theorem $(\forall X)\mathsf{CUT}(X)$.

Proposition 3.7. *Assume* $\mathsf{SVC}^+(S)$ *and* $\mathrm{cf}(\omega_1) = \omega_1$*. The following are equivalent:*

1. CUT*;*

2. $\mathsf{CUT}(S)$.

Proof. Certainly CUT implies $\mathsf{CUT}(S)$, so assume $\mathsf{CUT}(S)$. Let $\{A_n \mid n < \omega\}$ be a countable family of countable sets, and let $A = \bigcup_{n<\omega} A_n$, assuming without loss of generality that $A \subseteq S \times \eta$ is minitial. For $s \in S$, let $\eta_s = \{\alpha < \eta \mid \langle s, \alpha\rangle \in A\}$. Note that, for all s, η_s is a countable union of countable sets, and hence $\eta_s < \omega_1$. For $n < \omega$, let $B_n = \{s \in S \mid (\exists\alpha)\langle s, \alpha\rangle \in A_n\}$, so $B_n \subseteq S$ is countable for all n. By $\mathsf{CUT}(S)$, $B = \bigcup_{n<\omega} B_n$ is countable, and hence $|A| \leq |B \times \eta| \leq \aleph_0$ as required. □

Question 3.8. Can Proposition 3.7 be improved to "CUT is equivalent to $\mathsf{CUT}(S)$" without assuming that ω_1 is regular? Since the singularity of ω_1 is already a violation of CUT, this is equivalent to "does $\mathsf{SVC}^+(S)$ and $\mathrm{cf}(\omega_1) = \omega$ imply $\neg\mathsf{CUT}(S)$?".

3.4. **The axiom of choice.** The following was remarked by Blass in [1].

Proposition 3.9 (Blass)**.** *Assume* $\mathsf{SVC}(S)$*. The following are equivalent:*

1. AC*;*
2. *S can be well-ordered.*

Proof. Certainly AC implies that S can be well-ordered. On the other hand, if S can be well-ordered and $|X| \leq^* |S \times \eta|$ then $|X| \leq |S \times \eta|$, so X can be well-ordered. □

3.5. **Comparability.** W_X is the statement "for all Y, $|X| \leq |Y|$ or $|Y| \leq |X|$" and W^*_X is the statement "for all Y, $|X| \leq^* |Y|$ or $|Y| \leq^* |X|$". We write $\mathsf{W}^{(*)}_X(B)$ to mean "for all $A \subseteq B$, $|X| \leq^{(*)} |A|$ or $|A| \leq^{(*)} |X|$". Note that "every infinite set is Dedekind-infinite" is equivalent to $\mathsf{W}_{\aleph_0}$.

Proposition 3.10. *Assume* $\mathsf{SVC}^+(S)$*. The following are equivalent:*

1. W_λ*;*
2. $\mathsf{W}_\lambda(S)$.

Proof. Certainly W_λ implies $\mathsf{W}_\lambda(S)$, so assume $\mathsf{W}_\lambda(S)$. Let $X \subseteq S \times \eta$ be minitial, and let $A = \{s \in S \mid \langle s, 0\rangle \in X\}$. If $|A| \leq |\lambda|$ then X is well-orderable and so either $|X| \leq |\lambda|$ or $|\lambda| \leq |X|$. On the other hand, if $|A| \nleq |\lambda|$ then $|\lambda| \leq |A| \leq |X|$ as required. □

Replacing $\leq$ by $\leq^*$ in the proof of Proposition 3.10, we obtain Proposition 3.11.

Proposition 3.11. *Assume* $\mathsf{SVC}^+(S)$*. The following are equivalent:*

1. W^*_λ*;*
2. $\mathsf{W}^*_\lambda(S)$.

Question 3.12. As a consequence of Propositions 3.10 and 3.11, assuming $\mathsf{SVC}(S)$ (and hence $\mathsf{SVC}^+(\mathscr{P}(S))$), $\mathsf{W}_\lambda(\mathscr{P}(S))$ implies W_λ, and $\mathsf{W}^*_\lambda(\mathscr{P}(S))$ implies W^*_λ. Under the assumption of $\mathsf{SVC}(S)$, can we obtain a 'better' set X such that $\mathsf{W}_\lambda(X)$ implies W_λ? What about the W^*_λ case?

3.6. **Boolean prime ideal theorem.** The Boolean prime ideal theorem BPI is the statement "every Boolean algebra has a prime ideal", though it has many equivalent forms (see [2]). In [1], Blass presents the following local reflection of BPI under the assumption of SVC, attributing the proof to Pincus.

Proposition 3.13 (Pincus)**.** *Assume* $\mathsf{SVC}(S)$*. The following are equivalent:*

1. BPI*;*
2. *There is a fine ultrafilter on $[S]^{<\omega}$. That is, an ultrafilter $\mathcal{U}$ on $[S]^{<\omega}$ such that, for all $s \in S$, $\{a \in [S]^{<\omega} \mid s \in a\} \in \mathcal{U}$.*

3.7. **Kinna–Wagner principles.** For an ordinal α, KWP_α means "for all X there is an ordinal η such that $|X| \leq |\mathscr{P}^\alpha(\eta)|$", and KWP^*_α means "for all X, there is an ordinal η such that $|X| \leq^* |\mathscr{P}^\alpha(\eta)|$". The following observations, from [6], are consequences of the fact that, for all α, there is a definable surjection from $\mathscr{P}^\alpha(\mathrm{Ord})$ onto $\mathscr{P}^\alpha(\mathrm{Ord}) \times \mathrm{Ord}$, and that $\mathrm{Ord} \subseteq \mathscr{P}^\alpha(\mathrm{Ord})$.

Proposition 3.14 (Karagila–Schilhan)**.** *Assume* $\mathsf{SVC}(S)$*. The following are equivalent:*

1. KWP^*_α*;*
2. *There is* $\eta \in \mathrm{Ord}$ *such that* $|S| \leq^* |\mathscr{P}^\alpha(\eta)|$*.*

Proposition 3.15 (Karagila–Schilhan)**.** *Assume* $\mathsf{SVC}^+(S)$*. The following are equivalent:*

1. KWP_α*;*
2. *There is* $\eta \in \mathrm{Ord}$ *such that* $|S| \leq |\mathscr{P}^\alpha(\eta)|$*.*

Remark. Given that KWP_0 and KWP^*_0 are both equivalent to AC, Propositions 3.14 and 3.15 give new context to Proposition 3.9.

3.8. **The partition principle.** The partition principle PP says "for all X and Y, $|X| \leq |Y|$ if and only if $|X| \leq^* |Y|$". Note that the forward implication always holds. By $\mathsf{PP}\restriction X$ we mean the partition principle for subsets of X: If $A, B \subseteq X$ and $|A| \leq^* |B|$ then $|A| \leq |B|$. We instead write $\mathsf{PP}(X)$ to mean "for all A, if $|A| \leq^* |X|$ then $|A| \leq |X|$".

Proposition 3.16. *Assume* $\mathsf{SVC}^+(S)$*. The following are equivalent:*

1. PP*;*
2. $\mathsf{PP}\restriction S$ *and* $\mathsf{AC}_{\mathsf{WO}}$*.*

Proof. Certainly PP implies $\mathsf{PP}\restriction S$, and PP implies "for all X, $\aleph(X) = \aleph^*(X)$", which is equivalent to $\mathsf{AC}_{\mathsf{WO}}$. So instead assume $\mathsf{PP}\restriction S \wedge \mathsf{AC}_{\mathsf{WO}}$. Let $A, B \subseteq S \times \eta$ be such that $|A| \leq^* |B|$, witnessed by $f\colon B \to A$. We treat f as a partial surjection $f\colon S \times \eta \to A$. For $\langle t, \alpha\rangle \in A$, let

$$\varepsilon_{t,\alpha} = \min\{\varepsilon < \eta \mid (\exists s) f(s, \varepsilon) = \langle t, \alpha\rangle\}.$$

Let $B^{\langle t,\alpha\rangle} = \{s \in S \mid f(s, \varepsilon_{t,\alpha}) = \langle t, \alpha\rangle\}$, and $B^\varepsilon = \bigcup\{B^{\langle t,\alpha\rangle} \mid \varepsilon_{t,\alpha} = \varepsilon\}$. Let $E = \{\varepsilon < \eta \mid (\exists\langle t, \alpha\rangle)\varepsilon_{t,\alpha} = \varepsilon\} = \{\varepsilon < \eta \mid B^\varepsilon \neq \emptyset\}$. For each $\varepsilon \in E$, let $A^\varepsilon = \{\langle t, \alpha\rangle \mid \varepsilon_{t,\alpha} = \varepsilon\}$. Then $|A^\varepsilon| \leq^* |B^\varepsilon|$, witnessed by $s \mapsto f(s, \varepsilon)$. Hence, by assumption, $|A^\varepsilon| \leq |B^\varepsilon|$ and $I^\varepsilon = \{\text{injections } A^\varepsilon \to B^\varepsilon\} \neq \emptyset$. By $\mathsf{AC}_{\mathsf{WO}}$, let $c\colon E \to \bigcup\{I^\varepsilon \mid \varepsilon \in E\}$ be a choice function. Then

$$\begin{aligned} g\colon \quad & A \to S \times \eta \\ & \langle t, \alpha\rangle \mapsto \langle c(\varepsilon_{t,\alpha})(t, \alpha), \varepsilon_{t,\alpha}\rangle \end{aligned}$$

is an injection. Furthermore, for each $\langle t, \alpha\rangle \in A$, $f(g(t, \alpha)) \in A^{\varepsilon_{t,\alpha}}$. In particular, $f(g(t, \alpha))$ is defined and so g is in fact an injection $A \to B$ as required. $\square$

Question 3.17. Can Proposition 3.16 be improved to "PP is equivalent to $\mathsf{PP}\restriction S$"? Equivalently, is $\mathsf{AC}_{\mathsf{WO}}$ a consequence of $\mathsf{SVC}^+(S) \wedge \mathsf{PP}\restriction S$?

While we do not know if $\mathsf{AC}_{\mathsf{WO}}$ is unnecessary in Proposition 3.16, we cannot weaken the requirement of $\mathsf{SVC}^+(S)$ to $\mathsf{SVC}(S)$, as Proposition 3.18 demonstrates.

Proposition 3.18. *Let* M *be the Feferman-style model* $\mathfrak{N}_{\aleph_1}$ *from* [8]*. That is, for* $G \subseteq \mathrm{Add}(\omega, \omega_1)$ *an* L*-generic filter, we set*

$$M = L\left(\{\langle c_\beta \mid \beta < \alpha\rangle \mid \alpha < \omega_1\}\right),$$

where c_β *is the* βth *Cohen real introduced by* G*. Then*

$$M \vDash \mathsf{AC}_{\mathsf{WO}} \wedge \mathsf{PP}\restriction \mathbb{R} \wedge \mathsf{SVC}(\mathbb{R}) \wedge \neg\mathsf{PP}.$$

Proof. Firstly, by [8, Lemma 2.4], $M \vDash \mathsf{AC_{WO}}$.

By [8, Theorem 3.2], every set of reals in M can either be well-ordered or contains a perfect subset. If $A, B \subseteq \mathbb{R}$ and $|A| \leq^* |B|$ then: If B can be well-ordered, $|A| \leq |B|$; and if B contains a perfect subset then $|A| \leq |\mathbb{R}| \leq |B|$. Hence $\mathsf{PP}\restriction\mathbb{R}$ holds.

By [8, Lemma 2.3], $M \vDash V = L(w(\mathbb{R}))$, where $w(\mathbb{R})$ is the set of well-orders of subsets of $\mathbb{R}$. We aim to show that $w(\mathbb{R}) \subseteq L(\mathbb{R})$ and so $M \vDash V = L(\mathbb{R})$. In particular, this will show that $M \vDash \mathsf{SVC}(\mathbb{R})$.[1] As a consequence of [8, Theorem 3.1], a set X of reals in M is well-orderable if and only if $X \subseteq L[\langle c_\beta \mid \beta < \alpha\rangle]$ for some $\alpha < \omega_1$. Hence, any well-ordered sequence $f\colon \gamma \to \mathbb{R}$ in M is in fact an element of $L[\langle c_\beta \mid \beta < \alpha\rangle]$ some α. Since $\alpha < \omega_1$, we may encode the entire sequence $\langle c_\beta \mid \beta < \alpha\rangle$ as a single real c, and so $f \in L[c]$ for some $c \in \mathbb{R}$. Hence, $w(\mathbb{R}) \subseteq L(\mathbb{R}) \subseteq L(w(\mathbb{R}))$, and so $M \vDash V = L(\mathbb{R})$.

It remains to show that $M \vDash \neg\mathsf{PP}$, which we shall do this by showing that there is no injection $[\mathbb{R}]^\omega \to \mathbb{R}$ (noting that $\mathsf{ZF} \vdash |[\mathbb{R}]^\omega| \leq^* |\mathbb{R}|$).

Suppose that $F\colon [\mathbb{R}]^\omega \to \mathbb{R}$ is such an injection, and assume that it is L-definable. Let $\mathbb{P} = \mathrm{Add}(\omega, \omega_1)$ and, for $p \in \mathbb{P}$, we define the support of p, $\mathrm{supp}(p)$, to be $\{\beta < \omega_1 \mid (\exists n)\langle \beta, n\rangle \in \mathrm{dom}(q)\} \in [\omega_1]^{<\omega}$. Since F is definable in L, F has a $\mathbb{P}$-name $\dot{F}$ such that for all $\sigma \in \mathrm{Aut}(\mathbb{P})$, $\sigma\dot{F} = \dot{F}$.[2] For $\beta < \omega_1$, we define $\dot{c}_\beta = \{\langle p, \check{n}\rangle \mid p(\beta, n) = 1\}$, so $\dot{c}_\beta$ is a name for c_β. Given a permutation π of ω_1, define $\hat{\pi} \in \mathrm{Aut}(\mathbb{P})$ by $\hat{\pi}p(\pi(\alpha), n) = p(\alpha, n)$ for all $p \in \mathbb{P}$ and $\langle \alpha, n\rangle \in \omega_1 \times \omega$. Note that for such automorphisms, $\hat{\pi}\dot{c}_\beta = \dot{c}_{\pi(\beta)}$. Let $\dot{C} = \{\dot{c}_n \mid n < \omega\}^\bullet$.[3] Then $C = \dot{C}^G \in [\mathbb{R}]^\omega \cap M$. Let $p_0 \in \mathbb{P}$ be such that $p_0 \Vdash$ "$\dot{F}$ is an injection $[\dot{\mathbb{R}}]^\omega \to \dot{\mathbb{R}}$".

Suppose that for some $\mathbb{P}$-name $\dot{x}$ and some $p \leq p_0$, $p \Vdash \dot{F}(\dot{C}) = \dot{x}$. Suppose also that for some $q \leq p$ and some $n < \omega$, $q \Vdash \check{n} \in \dot{x}$.

Claim 3.18.1. $q\restriction \mathrm{supp}(p) \times \omega \Vdash \check{n} \in \dot{x}$.

Proof of Claim. We shall show that for all $r \leq q\restriction \mathrm{supp}(p) \times \omega$, $r \nVdash \check{n} \notin \dot{x}$. Let $r \leq q\restriction \mathrm{supp}(p) \times \omega$ be arbitrary. Then there is a permutation π of ω_1 such that $\pi``\omega = \omega$, $\pi$ fixes $\mathrm{supp}(p)$ pointwise, and $\mathrm{supp}(r) \cap \pi``\mathrm{supp}(q) \subseteq \mathrm{supp}(p)$ (noting that $\mathrm{supp}(p)$, $\mathrm{supp}(q)$, and $\mathrm{supp}(r)$ are all finite). Then $\hat{\pi}p = p$, $\hat{\pi}\dot{C} = \dot{C}$, $\hat{\pi}\dot{F} = \dot{F}$, and so $\hat{\pi}q \Vdash \dot{F}(\dot{C}) = \hat{\pi}\dot{x}$. However, since $\hat{\pi}q \leq \hat{\pi}p = p$, $\hat{\pi}q \Vdash \dot{F}(\dot{C}) = \dot{x}$ as well, and thus $\hat{\pi}q \Vdash \hat{\pi}\dot{x} = \dot{x}$. Furthermore, $\hat{\pi}q \parallel r$, and so $\hat{\pi}q \cup r \Vdash \check{n} \in \dot{x}$. Therefore, $r \nVdash \check{n} \notin \dot{x}$. Since no $r \leq q\restriction \mathrm{supp}(p) \times \omega$ forces $\check{n} \notin \dot{x}$, we must have that $q\restriction \mathrm{supp}(p) \times \omega \Vdash \check{n} \in \dot{x}$. $\dashv$

By the claim, we may assume that $\dot{x}$ is an $\mathrm{Add}(\omega, \mathrm{supp}(p))$-name.

Let $\beta, \beta' \in \omega_1 \setminus \mathrm{supp}(p)$ be such that $\beta < \omega$ and $\beta' \geq \omega$, and let π be the transposition $\begin{pmatrix}\beta & \beta'\end{pmatrix}$. Then $\hat{\pi}p = p$ and $\hat{\pi}\dot{x} = \dot{x}$, so $p \Vdash \dot{F}(\hat{\pi}\dot{C}) = \dot{x}$. However, $p \Vdash \dot{c}_{\beta'} \in \hat{\pi}\dot{C} \setminus \dot{C}$, and so $p \Vdash \hat{\pi}\dot{C} \neq \dot{C}$, contradicting that $p \Vdash$ "$\dot{F}$ is an injection".

In the case that F requires a real parameter, say c, we note that by the c.c.c. of $\mathrm{Add}(\omega, \omega_1)$, c has an $\mathrm{Add}(\omega, \alpha)$-name for some $\alpha < \omega_1$. By working in $L[G\restriction\alpha]$ rather than L, and noting that the quotient of $\mathrm{Add}(\omega, \omega_1)$ by $\mathrm{Add}(\omega, \alpha)$ is isomorphic to $\mathrm{Add}(\omega, \omega_1)$, the same result follows (using $\{\dot{c}_\beta \mid \alpha \leq \beta < \alpha + \omega\}^\bullet$ instead of C). $\square$

Even though we cannot improve Proposition 3.16 to $\mathsf{SVC}(S)$ as written, we can if we additionally assume the stronger axiom $\mathsf{PP}(S)$, rather than merely $\mathsf{PP}\restriction S$.

Proposition 3.19. $\mathsf{SVC}(S) \wedge \mathsf{PP}(S) \wedge \mathsf{AC_{WO}}$ *implies* $\mathsf{SVC}^+(S)$.

---

[1] If A is transitive and $M = V(A)$, where $V \vDash \mathsf{ZFC}$ then $M \vDash \mathsf{SVC}([A]^{<\omega})$ (see [1]). However, $\mathsf{ZF} \vdash |[\mathbb{R}]^{<\omega}| = |\mathbb{R}|$.

[2] In fact $\dot{F} = \sigma\dot{F}$ for all automorphisms σ of the Boolean completion of $\mathbb{P}$.

[3] That is, $\dot{C} = \{\langle \mathbb{1}, \dot{c}_n\rangle \mid n < \omega\}$.

Proof. Let A be set and $h\colon S \times \tau \to A$ be a partial surjection. We may assume that for all $a \in A$, there is unique $\alpha_a < \tau$ such that, for some $s \in S$, $h(s, \alpha_a) = a$. For $\alpha < \tau$, let $S_\alpha = \{s \in S \mid \langle s, \alpha\rangle \in \mathrm{dom}(h)\} = \bigcup\{S_a \mid \alpha_a = \alpha\}$. For each $\alpha < \tau$, $|A_\alpha| \leq^* |S|$ witnessed by the partial surjection $s \mapsto h(s, \alpha)$, and so $|A_\alpha| \leq |S|$. Using $\mathsf{AC}_{\mathsf{WO}}$, we may pick injections $i_\alpha\colon A_\alpha \to S$ for all $\alpha < \tau$, and thus form an injection $A \to S \times \tau$ by $a \mapsto \langle i_{\alpha_a}(a), \alpha_a\rangle$. Since A was arbitrary, $\mathsf{SVC}^+(S)$ follows. ☐

Corollary 3.20. *Assume* $\mathsf{SVC}(S)$*. The following are equivalent:*

1. PP*;*
2. $\mathsf{PP}\restriction S$*,* $\mathsf{PP}(S)$*, and* $\mathsf{AC}_{\mathsf{WO}}$*.*

Proof. Certainly PP implies each of $\mathsf{PP}\restriction S$, $\mathsf{PP}(S)$, and $\mathsf{AC}_{\mathsf{WO}}$, so instead assume $\mathsf{PP}\restriction S$, $\mathsf{PP}(S)$ and $\mathsf{AC}_{\mathsf{WO}}$. By Proposition 3.19, $\mathsf{SVC}^+(S)$ holds, and so by Proposition 3.16, PP holds. ☐

By Lemma 3.21, we cannot omit the $\mathsf{PP}\restriction S \wedge \mathsf{AC}_{\mathsf{WO}}$ requirement from Corollary 3.20.

Lemma 3.21. *Let $M = L(A)$ be Cohen's first model, where $L \subseteq M \subseteq L[G]$ for L-generic $G \subseteq \mathrm{Add}(\omega, \omega)$. Then*

$$M \vDash \mathsf{SVC}^+(\mathbb{R}) \wedge \mathsf{PP}(\mathbb{R}) \wedge \neg\mathsf{PP}.$$

Proof. For an overview of Cohen's first model and the proof of $\mathsf{SVC}^+(\mathbb{R})$, see [3, Sections 5.3 and 5.5]. Within is also a proof that there is an infinite Dedekind-finite set in M, contradicting $\mathsf{AC}_{\mathsf{WO}}$ (and hence PP).

The proof that $M \vDash \mathsf{PP}(\mathbb{R})$ is due to Elliot Glazer and Assaf Shani. Suppose that $f\colon \mathbb{R} \to X$ is a surjection and, using $\mathsf{SVC}^+(\mathbb{R})$, assume that $X \subseteq \mathbb{R} \times \eta$ for some minimal η. In $L[G]$, $2^{\aleph_0} = \aleph_1$, and f induces a surjection $\mathbb{R} \to \eta$. Therefore $\eta < \omega_2$. In M, $|\omega_1| \leq |\mathbb{R}|$, and so

$$|X| \leq |\mathbb{R} \times \eta| \leq |\mathbb{R} \times \omega_1| \leq |\mathbb{R} \times \mathbb{R}| \leq |\mathbb{R}|.$$ ☐

Furthermore, the model $\mathfrak{N}_{\aleph_1}$ from Proposition 3.18 shows that we cannot omit the $\mathsf{PP}\restriction S$ requirement from Corollary 3.20.

Lemma 3.22. *Let M be the Feferman-style model $\mathfrak{N}_{\aleph_1}$ from [8] (and Proposition 3.18). Then*

$$M \vDash \mathsf{AC}_{\mathsf{WO}} \wedge \mathsf{PP}(\mathscr{P}(\mathbb{R})) \wedge \mathsf{SVC}^+(\mathscr{P}(\mathbb{R})) \wedge \neg\mathsf{PP}.$$

Proof. We already saw in Proposition 3.18 that M is a model of $\mathsf{AC}_{\mathsf{WO}} \wedge \mathsf{SVC}(\mathbb{R}) \wedge \neg\mathsf{PP}$. By $\mathsf{SVC}(\mathbb{R})$, $\mathsf{SVC}^+(\mathscr{P}(\mathbb{R}))$ holds.

The proof of $\mathsf{PP}(\mathscr{P}(\mathbb{R}))$ is similar to Lemma 3.21. Suppose that $|X| \leq^* |\mathscr{P}(\mathbb{R})|$, witnessed by f. We may assume that $X \subseteq \mathscr{P}(\mathbb{R}) \times \eta$ for minimal η. Since the outer model $L[G] \vDash |\mathscr{P}(\mathbb{R})| = \aleph_2$, and f induces a surjection $\mathscr{P}(\mathbb{R})^M \to \eta$, we must have that $\eta < \omega_3$. In M, $|\omega_2| \leq |\mathscr{P}(\mathbb{R})|$ and so

$$|X| \leq |\mathscr{P}(\mathbb{R}) \times \eta| \leq |\mathscr{P}(\mathbb{R}) \times \omega_2| \leq |\mathscr{P}(\mathbb{R})^2| \leq |\mathscr{P}(\mathbb{R}^2)| \leq |\mathscr{P}(\mathbb{R})|.$$ ☐

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