Proper classes of maximal θ-independent families

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This zine is a heavily abridged version of [10]. In particular, the online version contains more preliminaries, exposition, exploration of the literature, details, depth, and improved typesetting. Many shorthands have been taken that are not usual mathematical practice in order to save space.

Abstract

While maximal independent families can be constructed in ZFC via Zorn's lemma, the presence of a maximal σ-independent family already gives an inner model with a measurable cardinal, and Kunen has shown that from a measurable cardinal one can construct a forcing extension in which there is a maximal σ-independent family. We extend this technique to construct proper classes of maximal θ-independent families for various uncountable θ. In the first instance, a single θ^+-strongly compact cardinal has a set-generic extension with a proper class of maximal θ-independent families. In the second, we take a class-generic extension of a model with a proper class of measurable cardinals to obtain a proper class of θ for which there is a maximal θ-independent family.

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1 Introduction

[8] exhibits the equiconsistency of a measurable cardinal and a maximal σ-independent family. For infinite θ and infinite X, $\mathcal{A} \subseteq \mathscr{P}(X)$ is *θ-independent* if $|\mathcal{A}| \geq \theta$ and for all partial $p\colon \mathcal{A} \to 2$ with $|p| < \theta$,

$$\mathcal{A}^p := \bigcap\{A \mid p(A) = 1\} \cap \bigcap\{X \setminus A \mid p(A) = 0\} \neq \emptyset.$$

σ-independent means $\aleph_1$-independent. A maximal θ-independent family (*MθIF*) is a θ-independent family maximal among θ-independent families by inclusion. By Zorn's Lemma, ZFC proves the existence of MωIFs, but MσIFs entail an inner model with a measurable cardinal, a fascinating increase in consistency strength.

In [8] Kunen comments that a single strongly compact cardinal κ would beget, in a forcing extension, MσIFs $\mathcal{A} \subseteq \mathscr{P}(\lambda)$ for all λ s.t. $\mathrm{cf}(\lambda) \geq \kappa$. We shall prove this and reduce the requirement to κ being $\aleph_1$-strongly compact, generalising to θ-independence. We also extend the technique to a proper class of measurables, iterating the process. In this model, the Mitchell rank of cardinals is very nearly preserved.

Preliminaries

Given a filter $\mathcal{F}$ on X, $\mathcal{F}^* := \{X \setminus A \mid A \in \mathcal{F}\}$. For $V \subseteq W$ models of ZFC and an ideal $\mathcal{I} \in V$ on X, $\langle \mathcal{I} \rangle^W := \{A \in \mathscr{P}(X)^W \mid (\exists Y \in \mathcal{I})\, A \subseteq Y\}$. An inner model M is *λ-closed* if $M^{<\lambda} \subseteq M$, or λ-closed *in* V to emphasise $M^{<\lambda} \cap V \subseteq M \subseteq V$. Our convention for the Mitchell order ([9]) is that $o(\kappa) > 0$ if and only if κ is measurable. Following [1], for $\theta \leq \kappa$, κ is *θ-strongly compact* if every κ-complete filter on any X can be extended to a θ-complete ultrafilter on X.

Theorem 1.1 ([1])**.** *TFAE:* (i) *κ is θ-strongly compact.*
(ii) $(\forall \alpha \geq \kappa)(\exists j\colon V \to M)$ *s.t.* $\mathrm{crit}(j) \geq \theta$ *and* $(\exists D \in M)(j``\alpha \subseteq D \wedge M \vDash |D| < j(\kappa))$.
(iii) $(\forall \alpha \geq \kappa)\exists$ *fine θ-complete u.f. on* $\mathscr{P}_\kappa(\alpha)$.

See [6] for information on forcing. The following is a combination and weakening of Lemma 13 and Theorem 10 in [5].

Proposition 1.2 (Hamkins)**.** *If* $\mathrm{Add}(\omega,1)$ *forces that* $\dot{\mathbb{Q}}$ *is* σ*-closed then for all* V*-generic* $G \subseteq \mathrm{Add}(\omega,1) * \dot{\mathbb{Q}}$ *and normal measures* $\mathcal{U} \in V[G]$ *on* κ*,* $\mathcal{U} \cap V \in V$ *is a normal measure on* κ*.*

2 Hammers

We have two tools for constructing MθIFs. The first is almost [8, Lemma 2.1], but we have softened the requirements.

Lemma 2.1. *Let* $\theta > \omega$ *be regular,* $|X| \geq \theta$*, and* $\mathcal{I}$ *a* θ*-complete ideal over* X *s.t.* $\mathrm{Add}(\theta, 2^X)$ *densely embeds into* $\mathscr{P}(X)/\mathcal{I}$*. Then there is an M*$\theta$*IF* $\mathcal{A} \subseteq \mathscr{P}(X)$*.*

The second is an old technique present in [8] (among other places) for obtaining these ideals. The version we present is a simplified form of the Duality Theorem ([2]).

Theorem 2.2. *Let* $\mathcal{U}$ *be a* σ*-complete ultrafilter on* X *with ultrapower embedding* $j\colon V \to M$*. Let* $\mathbb{P}$ *a forcing s.t.* $\pi\colon j(\mathbb{P}) \cong \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ *satisfying* $\pi(j(p)) = \langle p, \mathbb{1}, \mathbb{1}\rangle$*. If, for all* V*-generic* $G * H \subseteq \mathbb{P} * \dot{\mathbb{Q}}$*, there is* $M[G*H]$*-generic* $F \subseteq \dot{\mathbb{R}}^{G*H}$ *in* $V[G*H]$*, then there is an ideal* $\mathcal{I}$ *on* X *in* $V[G]$ *s.t.* $\mathscr{P}(X)/\mathcal{I} \cong B(\dot{\mathbb{Q}}^G)$*.*

3 Nails

The following is an extension of [8, Theorem 2], but the technique is essentially the same.

Theorem 3.1 ([8, Theorem 2])**.** *Let* κ *be a measurable cardinal,* $\theta \in (\omega, \kappa)$ *be regular, and* G *be* V*-generic for* $\mathrm{Add}(\theta,\kappa)$*. Then there is an M*$\theta$*IF* $\mathcal{A} \subseteq \mathscr{P}(2^\theta)$ *in* $V[G]$*.*

Kunen also recovers a measurable cardinal from an MθIF.

Theorem 3.2 ([8, Theorem 1])**.** *Let* $\theta > \omega$ *regular, and* $\mathcal{A} \subseteq \mathscr{P}(\lambda)$ *an M*θ*IF. Then* $2^{<\theta} = \theta$ *and, for some* κ *s.t.* $\sup\{(2^\alpha)^+ \mid \alpha < \theta\} \leq \kappa \leq \min\{\lambda, 2^\theta\}$*, there is a non-trivial* θ^+*-saturated* κ*-complete ideal over* κ*.*

Corollary 3.3 (Kunen)**.** *If there is an MθIF for regular $\theta > \omega$ then there is an inner model containing a measurable cardinal.*

Kunen sketches how to obtain an MκIF on inaccessible κ, starting with measurable κ. We also include additional content about lifting normal measures, which we use for Theorem B.

Proposition 3.4. *Let $\mathcal{U}$ be a normal measure on κ, $2^\kappa = \kappa^+$, $A \in \mathcal{U}$ a set of regular cardinals, and G be V-generic for the Easton-support iteration $\mathbb{P} = \ast_{\alpha\in A} \mathrm{Add}(\alpha, \alpha^+)$. In $V[G]$ there is an MκIF $\mathcal{A} \subseteq \mathscr{P}(\kappa)$. If $\mathcal{V} \in V$ is a normal measure on κ s.t. $A \notin \mathcal{V}$ then there is normal $\hat{\mathcal{V}} \supseteq \mathcal{V}$ on κ in $V[G]$.*

Sketch proof. Let $j = j_{\mathcal{U}} \colon V \to M$, and $H \subseteq \mathrm{Add}(\kappa, \kappa^+)$ be $V[G]$-generic. Both M and $M[G * H]$ are κ^+-closed. Let $\dot{\mathbb{R}} = j(\mathbb{P})/(\mathbb{P}*\mathrm{Add}(\kappa, \kappa^+))$. Then $j(\mathbb{P}) \cong \mathbb{P}*\mathrm{Add}(\kappa, \kappa^+)*\dot{\mathbb{R}}$ as required in Theorem 2.2. Let

$$\mathbb{R} = \dot{\mathbb{R}}^{G*H} = \mathop{\ast}_{\alpha\in j(A)\setminus\kappa^+} \mathrm{Add}(\alpha, (\alpha^+)^M)^M.$$

$|\mathbb{P}|^V = \kappa$, so $|j(\mathbb{P})|^M = j(\kappa)$. Hence $|\mathbb{R}|^{V[G]} = \kappa^+$. Each iterand of $\mathbb{R}$ is α-closed according to M for some $\alpha \geq \kappa^+$, so each iterand of $\mathbb{R}$ is κ^+-closed (in $V[G * H]$) and, since it is an iteration of length $j(\kappa) \geq \kappa^+$, $\mathbb{R}$ itself is κ^+-closed. $\mathbb{P}$ has only κ-many maximal antichains, so $M[G * H] \vDash$ "$\mathbb{R}$ has only $j(\kappa)$ maximal antichains". $j(\kappa) < \kappa^{++}$, so we can build an $M[G * H]$-generic filter $F \subseteq \mathbb{R}$ in $V[G * H]$. By Theorem 2.2, in $V[G]$ there is $\mathcal{I}$ on κ s.t. $\mathscr{P}(\kappa)/\mathcal{I} \cong B(\mathrm{Add}(\kappa, \kappa^+))^{V[G]}$. This $\mathcal{I}$ is κ-complete, so Lemma 2.1 gives an MκIF on κ in $V[G]$.

On the other hand, let $\mathcal{V} \in V$ be normal on κ with $A \notin \mathcal{V}$ and embedding $i \colon V \to N$. $i(\mathbb{P}) \cong \mathbb{P} * \dot{\mathbb{R}}$, so $F \in V[G]$ and we may lift i to $\hat{\imath} \colon V[G] \to M[G * F]$ in $V[G]$ to get $\hat{\mathcal{V}} \supseteq \mathcal{V}$. □

A θ^+-strongly compact cardinal

Theorem A. *Let κ be θ^+-strongly compact for regular $\theta \in (\omega, \kappa)$, with $2^{<\kappa} = \kappa$, and let G be V-generic for $\mathrm{Add}(\theta, \kappa)$. In $V[G]$, for all λ with $\mathrm{cf}(\lambda) \geq \kappa$, there is an M$\theta$IF $\mathcal{A} \subseteq \mathscr{P}(\lambda)$.*

Proof. Let $\mathrm{cf}(\lambda) \geq \kappa$. We use Lemma 2.1 to find an MθIF $\mathcal{A} \subseteq \mathscr{P}(X)$, where $X = \mathscr{P}_\kappa(\lambda)^V$ (so $|X| = \lambda$). First we use Theorem 2.2 to get a θ-complete ideal $\mathcal{I}$ over X s.t. $B(\mathrm{Add}(\theta, 2^X)) \cong \mathscr{P}(X)/\mathcal{I}$ in $V[G]$: Let $\mathcal{U} \in V$ be a fine θ-complete ultrafilter on X and $j = j_{\mathcal{U}} \colon V \to M$. Since $\kappa \geq \mathrm{crit}(j) > \theta$,

$$\begin{aligned} j(\mathrm{Add}(\theta,\kappa)) &= \mathrm{Add}(\theta, j(\kappa)) \\ &\cong \mathrm{Add}(\theta, j``\kappa) \times \mathrm{Add}(\theta, j(\kappa) \setminus j``\kappa) \\ &\cong \mathrm{Add}(\theta,\kappa) \times \mathrm{Add}(\theta, j(\kappa) \setminus \kappa) \times \{\mathbb{1}\}. \end{aligned}$$

Each $p \in \mathrm{Add}(\theta,\kappa)$ is s.t. $|p| < \theta$, so $j(p) = j``p$, i.e. the isomorphism extends $j(p) \mapsto \langle p, \mathbb{1}, \mathbb{1}\rangle$. Setting $\mathcal{I} = \langle \mathcal{U}^* \rangle^{V[G]}$, we have $B(\mathrm{Add}(\theta, j(\kappa) \setminus \kappa)^V) \cong \mathscr{P}(X)/\mathcal{I}$ in $V[G]$ by Theorem 2.2. To finish we must show that $\mathrm{Add}(\theta, j(\kappa)\setminus\kappa)^V \cong \mathrm{Add}(\theta, 2^X)^{V[G]}$.

$\mathrm{Add}(\theta,\kappa)$ is θ-closed so we have, for all $Y \in V$, $\mathrm{Add}(\theta, Y)^V = \mathrm{Add}(\theta, Y)^{V[G]}$, so it is sufficient to show that $|j(\kappa)\setminus\kappa| = |(2^\lambda)^{V[G]}|$.

By standard chain condition techniques[1] and some cardinal arithmetic, we have $|(2^\lambda)^{V[G]}| \leq |(2^\lambda)^V|$. $(2^\lambda)^V \subseteq (2^\lambda)^{V[G]}$, so $|(2^\lambda)^V| = |(2^\lambda)^{V[G]}|$. Now we need only show that $|2^\lambda| = |j(\kappa)\setminus\kappa|$ *in* V. We work in V for the rest of the proof.

$2^\lambda > \kappa$ so it is enough to show that $2^\lambda \leq j(\kappa) < (2^\lambda)^+$. Let $D = [\mathrm{id}]_{\mathcal{U}}$ in M. By the fineness of $\mathcal{U}$, $j``\lambda \subseteq D$ and $M \vDash |D| < j(\kappa)$. By elementarity, $M \vDash (\forall \gamma < j(\kappa))2^\gamma \leq j(\kappa)$ and hence $M \vDash |\mathscr{P}(D)^M| \leq j(\kappa)$. Hence, by [4], $2^\lambda \leq |\mathscr{P}(D)^M|$.[2]

On the other hand, $j(\kappa) = \{[f]_{\mathcal{U}} \mid f \colon X \to \kappa\}$, so $j(\kappa) < (\kappa^\lambda)^+ = (2^\lambda)^+$. Thus $2^\lambda \leq j(\kappa) < (2^\lambda)^+$ as required. □

A class of measurable cardinals

Assume GCH and let $\kappa < \lambda$ be measurable. If $G \subseteq \mathbb{P}$ is V-generic and $|\mathbb{P}| < \lambda$, then λ is measurable in $V[G]$. Via Proposition 3.4, forcing with $\mathbb{P}_0 = \ast_{\alpha \in A} \mathrm{Add}(\alpha, \alpha^+)$, where $A = \{\alpha < \kappa \mid \alpha \text{ regular}\}$, gives an M$\kappa$IF $\mathcal{A} \subseteq \mathscr{P}(\kappa)$. $|\mathbb{P}_0| = \kappa$, so λ is still measurable in the extension. Repeating with

[1]One could adapt the proof of [6, Lemma 15.1] to incorporate chain conditions, for example.

[2]The method is similar to [7, Lemma 3.3.2], but could be older.

$\mathbb{P}_1 = \ast_{\alpha\in A'} \mathrm{Add}(\alpha,\alpha^+)$ in the extension, where $A' = \{\alpha < \lambda \mid \kappa < \alpha \wedge \alpha \text{ regular}\}$, Proposition 3.4 gives an MλIF $\mathcal{A}' \subseteq \mathscr{P}(\lambda)$ in the extension by $\mathbb{P}_1$. Since $\mathbb{P}_1$ is κ^+-closed, $\mathcal{A}$ is still an MκIF in the second extension. We continue iterating this way to produce a (class-size) forcing extension $V[G]$ s.t. whenever $o(\kappa)^V > 0$ there is an MκIF on κ in $V[G]$. The naïve attempt at this produces the Easton-support iteration $\ast_{\alpha\in A} \mathrm{Add}(\alpha,\alpha^+)$, where $A = \{\alpha \mid o(\alpha) = 0 \wedge \alpha \text{ regular}\}$ and hopes to use Proposition 3.4 to show that if $G \subseteq \mathbb{P}$ is V-generic and $\mathcal{U}$ is a normal measure on κ then there is an MκIF on κ in $V[G]$. This mostly works, but one must take care to ensure that $A \cap \kappa \in \mathcal{U}$. Any $\mathcal{U}$ with $o(\mathcal{U}) = 0$ satisfies this condition. For technical reasons, we also exclude successors of measurables from A.

Conversely, if $\mathcal{U} \in V$ and $o(\mathcal{U})^V > 0$ then there is $\hat{\mathcal{U}} \supseteq \mathcal{U}$ in $V[G]$ by a lifting argument. I.e., if $o(\kappa)^V > \alpha$ then $o(\kappa)^{V[G]} \geq \alpha$. In fact, we shall use a closure point argument à la Proposition 1.2 to get that if $o(\kappa)^V > 0$ then $o(\kappa)^V = 1 + o(\kappa)^{V[G]}$ exactly. Let use introduce some ad-hoc notation. For $\alpha \in \mathrm{Ord}$, let

$$^{-}\alpha := \begin{cases} 0 & \alpha = 0 \\ \alpha - 1 & 0 < \alpha < \omega \\ \alpha & \omega \leq \alpha. \end{cases}$$

So we will show that for all κ, $o(\kappa)^{V[G]} = {}^{-}o(\kappa)^V$.

Theorem B. *Let $V \vDash \mathsf{ZFC} + \mathsf{GCH}$. There is a class-length iteration $\mathbb{P}$ preserving $\mathsf{ZFC} + \mathsf{GCH}$ s.t. if $G \subseteq \mathbb{P}$ is V-generic, then whenever $o(\kappa)^V > 0$ there is an MκIF $\mathcal{A} \subseteq \mathscr{P}(\kappa)$ in $V[G]$. Also, for all κ, $o(\kappa)^{V[G]} = {}^{-}o(\kappa)^V$.*

Sketch proof. Let us define the iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \in \mathrm{Ord}\rangle$. $\mathbb{P}_\omega = \{\mathbb{1}\}$ and $\dot{\mathbb{Q}}_\omega = \mathrm{Add}(\omega, 1)$.[3] For $\alpha > \omega$, $\dot{\mathbb{Q}}_\alpha = \{\mathbb{1}\}$ if α is not a cardinal, is singular, is measurable, or is the successor of a measurable cardinal. Otherwise, let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\mathrm{Add}(\alpha,\alpha^+)$ in the extension. Iterate with Easton support. Let

[3] The ωth stage is different to apply a closure point argument later.

$\mathbb{P}$ be the direct limit of all $\mathbb{P}_\alpha$. By standard forcing techniques, $\mathbb{P}$ is tame and thus preserves ZFC.[4]

For measurable κ, $\mathbb{P} = \mathbb{P}_\kappa * (\mathbb{P}/\mathbb{P}_\kappa)$ with $\mathbb{P}/\mathbb{P}_\kappa$ forced to be κ^{++}-closed, so if $\mathbb{P}_\kappa$ adds an MκIF $\mathcal{A} \subseteq \mathscr{P}(\kappa)$ then it will still be an MκIF after forcing with $\mathbb{P}/\mathbb{P}_\kappa$. Picking normal $\mathcal{U} \in V$ s.t. $o(\mathcal{U})^V = 0$, apply Proposition 3.4 to find such a family.

We now show $(\forall\kappa)$, $o(\kappa)^{V[G]} = {}^{-}o(\kappa)^V$. $\mathbb{P}_{\kappa^{++}} = \mathrm{Add}(\omega, 1) * (\mathbb{P}_{\kappa^{++}}/\mathbb{P}_\omega)$, where $\mathbb{P}_{\kappa^{++}}/\mathbb{P}_\omega$ is σ-closed, so $\mathbb{P}_{\kappa^{++}}$ has a closure point at ω. Hence, if $\mathcal{U} \in V[G \restriction \kappa^{++}]$ is a normal measure on κ then, by Proposition 1.2, $\mathcal{U} \cap V \in V$. $\mathbb{P}/\mathbb{P}_{\kappa^{++}}$ is κ^{++}-closed, so any normal measure on κ in $V[G]$ was already in $V[G \restriction \kappa^{++}]$. Thus, if $o(\kappa)^{V[G]} > 0$ then $o(\kappa)^V > 0$, and so $\mathbb{P}_{\kappa^{++}} = \mathbb{P}_\kappa$.

Fact 3.4.1. *If $\mathcal{U} \in V[G]$ is a normal measure on κ, then $C_\kappa := \{\lambda < \kappa \mid o(\lambda)^V > 0\} \in \mathcal{U}$.*

We finally show $(\forall\kappa) o(\kappa)^{V[G]} = {}^{-}o(\kappa)^V$ by induction, so assume $(\forall\lambda < \kappa) o(\lambda)^{V[G]} = {}^{-}o(\lambda)^V$. $o(\kappa)^V = 0$ implies $o(\kappa)^{V[G]} = 0$, so assume $o(\kappa)^V > 0$.

$(o(\kappa)^{V[G]} \le {}^{-}o(\kappa)^V)$. Suppose $o(\kappa)^{V[G]} > {}^{-}o(\kappa)^V$, witnessed by $\mathcal{U} \in V[G]$ s.t. $o(\mathcal{U})^{V[G]} = {}^{-}o(\kappa)^V$. By Fact 3.4.1, $C_\kappa \in \mathcal{U} \cap V$, so $o(\mathcal{U} \cap V)^V > 0$ and $o(\kappa)^V > 1$, i.e. if ${}^{-}\alpha = {}^{-}o(\kappa)^V$ then $\alpha = o(\kappa)^V$. Thus,

$$\begin{aligned}\{\lambda < \kappa \mid o(\lambda)^{V[G]} = {}^{-}o(\kappa)^V\} &= \{\lambda < \kappa \mid {}^{-}o(\lambda)^V = {}^{-}o(\kappa)^V\} \\ &= \{\lambda < \kappa \mid o(\lambda)^V = o(\kappa)^V\} \in \mathcal{U} \cap V,\end{aligned}$$

so $o(\mathcal{U} \cap V)^V = o(\kappa)^V$, a contradiction.

$(o(\kappa)^{V[G]} \ge {}^{-}o(\kappa)^V)$. By Proposition 3.4, if $\mathcal{U} \in V$ is a normal measure on κ s.t. $A \cap \kappa \notin \mathcal{U}$ (i.e. $o(\mathcal{U})^V > 0$), there is $\hat{\mathcal{U}} \supseteq \mathcal{U}$ a normal measure in $V[G \restriction \kappa]$. $\mathbb{P}/\mathbb{P}_\kappa$ is κ^{++}-closed, so $\hat{\mathcal{U}}$ is a normal measure in $V[G]$. $o(\kappa)^{V[G]} \ge 0$, so assume $o(\kappa)^V > 1$. If $\mathcal{U} \in V$ is s.t. $o(\mathcal{U})^V > 0$ then

$$\{\lambda < \kappa \mid o(\lambda)^V = o(\mathcal{U})^V\} = \{\lambda < \kappa \mid o(\lambda)^{V[G]} = {}^{-}o(\mathcal{U})^V\} \in \hat{\mathcal{U}}.$$

So $(\forall\alpha < o(\kappa)^V) {}^{-}\alpha < o(\kappa)^{V[G]}$, i.e. ${}^{-}o(\kappa)^V \le o(\kappa)^{V[G]}$. □

[4] [6, Ch. 15] provides an overview of preservation of ZFC with class products. [3] has a treatment of class length iterations.

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